Nash Equilibrium

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- A combination of strategies such that every player is best-responding to every other player
- Can be pure or mixed, unique or or not
- A **pure strategy Nash Equilibrium** of G is an action profile $a \in A$ such that $u^i(a^i, a^{-i}) \geq u^i(\hat{a}^i, a^{-i})$ for all $\hat{a}^i \in A^i$

NE in experiments

Why should we expect NE to emerge in experiments?

1. Introspection
2. Learning (seeing the outcome of the game and playing again)

In some games with a unique equilibrium, its reasonable to expect learning

Experimental Protocols

Experiments typically conducted using one of the following protocols:

1. One-shot
    - Introspection
2. Strangers (random re-matching)
    - One-shot game with learning
    - If no feedback, learning is based on introspection
3. Partners (re-matched with the same partner over and over)
    - One-shot game becomes a repeated game
    - **Possibly predictions** (more on repeated games later)

Prisoner's Dilemma

Simplest, most famous game:

	C	D
C	7, 7	0, 12
D	12, 0	4, 4

- (D, D) is the unique Nash Equilibrium
- In fact, there is a strictly dominant strategy to defect for both players (C is not rational)

Andreoni and Miller (2003)

Andreoni and Miller conduct a PD experiment with four treatments, 14 subjects per session:

- **Partners:** 14 subjects randomly matched to play 10 rounds of PD in a fixed pair. Process repeated 20 times.
- **Strangers:** 14 subjects randomly matched to play 1 rounds of PD in a fixed pair. Process repeated 200 times.
- **Computer50:** 14 subjects randomly matched to play 10 rounds of PD in a fixed pair. 50% chance of the other player being a computer. Process repeated 20 times.
- **Computer0:** 14 subjects randomly matched to play 10 rounds of PD in a fixed pair. 0.1% chance of the other player being a computer. Process repeated 20 times.

Computer plays tit-for-tat: begin by cooperating, play what opponent played in the previous round

Not equilibrium, but best response is to cooperate for nine rounds

Predictions

- In the **strangers** treatment, subjects are playing a sequence of "one-shot" games. The prediction is that they defect in every period.
- ...Unless they are altruists
- A model of reciprocal altruism:

	C	D
C	$7+\alpha$, $7+\alpha$	0, 12
D	12, 0	4, 4

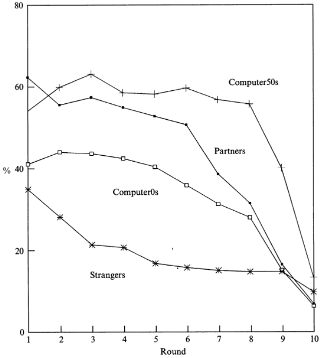
Predictions

- In the **partners** treatment, cooperation can be sustained in the first few periods by a reputation-building model
- For example, if both players assign a small probability $\delta > 0$ to facing a tit-for-tat player or a reciprocal altruist, an equilibrium of the finitely repeated game can be constructed where cooperation is observed in the first few periods
- **This is true even if no altruistic types exist!** Players just have to believe they do

Predictions

- The **Computer50** treatment is designed to test the reputation-building model
- Beliefs about cooperative types matter $\Rightarrow$ increased cooperation rates relative to **Partner**
- No effect would contradict the reputation-building model
- The **Computer0** treatment is designed to control for knowledge of tit-for-tat strategy. If no difference between **Partners** and **Computer0** but difference between **Partners** and **Computer50**, additional evidence of reputation-building

Results



- Cooperation gets pretty close to zero by period 10 of the strangers treatment! Though authors claim no significant period trend...
- Strange effect of partner vs. Computer0, but not significant
- Partners cooperate more than strangers ⇒ evidence of reputation-building
- Computer50 more cooperative than partners ⇒ additional evidence of reputation-building

Cooper, et. al (1996)

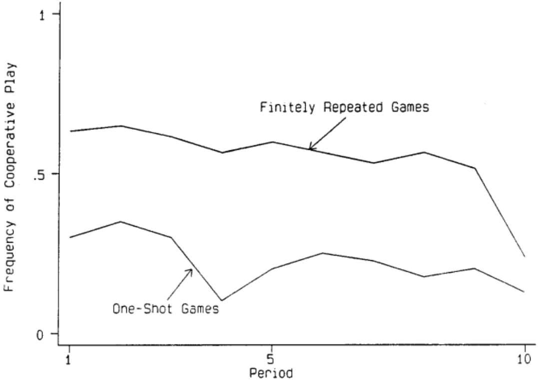
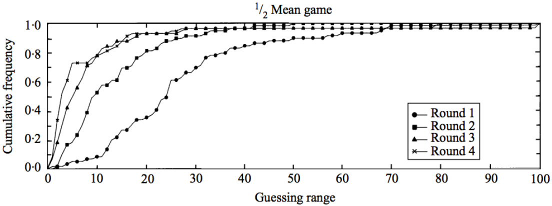


FIGURE 1

One-shot PD

- In both AM (1993) and Cooper, et al (1996), cooperation rates in the strangers treatment get close to zero over time
- While the authors interpret positive cooperation as altruism, a lot of it is probably due to noise
- Over time, the unique NE prediction fares reasonably well
- Remember we also saw this in beauty contest games:



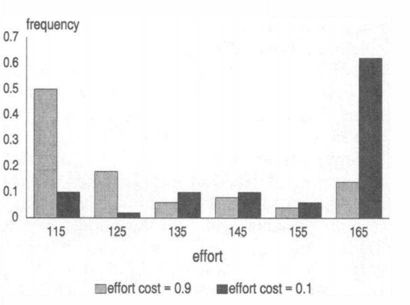
- **Observation:** When NE is unique, behavior in experiments often converges to it

Weakest link game 1

- $i \in \{1, 2\}$
- $e^i \in \{110, 111, ..., 170\}$ for all i
  - e for "effort"
- $u^i(e^i, e^{-i}) = min(e^i, e^{-i}) - ce^i$
- $0 < c < 1$
- **Let's do a classroom experiment!**

Weakest link game 1

- Results in Goeree and Holt (2005):

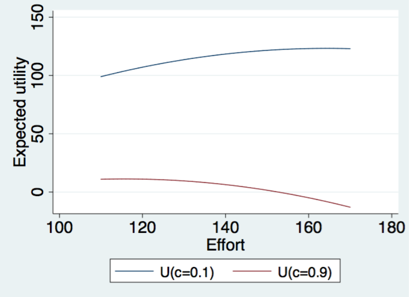


- Nash predictions? Any combination efforts with $e^i = e^{-i}$ is a Nash equilibrium.
- **Important role for experiments:** If a game has multiple equilibria, an experiment can allow us to see which one is selected
- Contrast this with experiments testing for unique equilibrium

A simple explanation

- A Level-1 model fits the story
- If $-i$ chooses uniformly,

$$E(u(e^i)) = \frac{1}{61}\sum_{e^i \le e^{-i}} e^i + \frac{1}{61}\sum_{e^i > e^{-i}} e^{-i} - ce^i$$



Why should we use a Level-1 model?

- **Principle of Insufficient Reason (Bernoulli, Laplace):** Assign same probabilities to events unless there is a reason to assign different ones
- Think of each possible action of the opponent as an event
- In a one-shot setting with every action an equilibrium, reasonable to assign equal probabilities to all actions

Experiments on equilibrium selection

- Let's look at some classic ones... Many of these experiments make use of **stag hunt games**

- A stag hunt game is a game with two pure strategy Nash Equilibria, one payoff-dominant and one risk-dominant:

	Stag	Hare
Stag	13, 13	-7, 10
Hare	10, -7	3, 3

- (Stag, Stag) is a **payoff-dominant** equilibrium: it Pareto dominates all other equilibria

- (Hare, Hare) is a **risk-dominant** equilibrium: in a 2x2 symmetric game, a risk-dominant equilibrium is a best response to a Laplacian prior

- Risk- and payoff-dominance are equilibrium refinements (selection principles) introduced by Harsanyi and Selten

Experiments on equilibrium selection

	Stag	Hare
Stag	13, 13	-7, 10
Hare	10, -7	3, 3

- Note that the **risk-dominant equilibrium** has higher **basin of attraction**
- *BasinStag*=Probability player must assign to the other player playing *Hare* that makes player indifferent between *Hare* and *Stag*
- If this number is small, *Hare* is attractive for a large range of beliefs about other player
- In experiments, behavior tends to converge to the risk-dominant equilibrium
- This represents **coordination failure** because players fail to coordinate on the payoff-dominant equilibrium

Experiments on equilibrium selection

Another weakest link game (Van Huyck, et al (1990)):

- Some number $N > 2$ of players (typically large)
- $u^i(e^1, e^2, ..., e^N) = 0.2min\{e^1, e^2, ..., e^N\} - 0.1e^i + 0.6$
- **Table 2** Earnings table for the "Minimum game" (Table A in VHBB, 1990)

	Smallest value of *X* chosen						
Your choice of *X*	7	6	5	4	3	2	1
7	1.30	1.10	0.90	0.70	0.50	0.30	0.10
6	–	1.20	1.00	0.80	0.60	0.40	0.20
5	–	–	1.10	0.90	0.70	0.50	0.30
4	–	–	–	1.00	0.80	0.60	0.40
3	–	–	–	–	0.90	0.70	0.50
2	–	–	–	–	–	0.80	0.60
1	–	–	–	–	–	–	0.70

- **Payoff-dominant** equilibrium is (7,7,7,7,...)
- No notion of risk-dominance... but **secure** equilibrium is (1,1,1,1,...)
- Formalize secure action as maximin$=max_{e^i} min_{e^{-i}} u^i(e)$

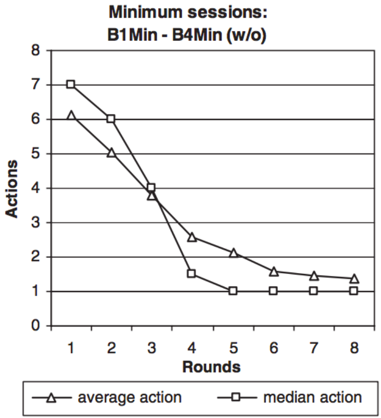
Uncertainty creates desire for security

- Why would players coordinate on the secure equilibrium?
- Assume player i has cdf $F(e^j)$ about effort of any other player
- If $e^1, e^2, e^3, ..., e^n$ are i.i.d. with $F(e^j)$, belief about the minumum follows

$$F_{min}(e) = 1 - (1 - F(e))^n$$

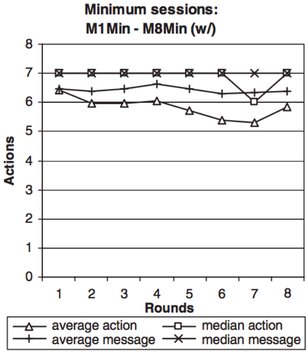
- Assume $F(1) > 0$. Then $F_{min}(1) \to 1$ as $n \to \infty$.

Blume and Ortmann (2007)



Overcoming coordination failure

- Many coordination games (stag hunt, weakest link) lead to **coordination failure**
- How can we influence subjects to choose the efficient equilibrium?
- Large literature on this
- One common intervention: communication



More on communication later...

	L	R
Top	80, 40	40, 80
Bottom	40, 80	80, 40

https://linkto.run/p/V52VWJ34

	L	R
Top	320, 40	40, 80
Bottom	40, 80	80, 40

https://linkto.run/p/KJK5FSBT

Results

- Version 1: `https://www.poll-maker.com/results2554305xC0AB25A6-73`
- Version 2: `https://www.poll-maker.com/results2554311xacFeF1DF-73`

Mixed Strategies

- Consider a two player game
- In a mixed strategy NE, the mixing probabilities of player i are determined by the **other** player's payoffs
- 1 mixes to make player 2 indifferent, which means 2's payoffs affect 1's mixing probabilities
- Same story for 2
- (Why must players be indifferent between their actions? If not indifferent, mixing is not a best response!)
- Letting $u^1(T, L) = x$, the mixing probabilities are $(1/2, 1, 2)$ for player 1 and $(\frac{40}{x}, 1 - \frac{40}{x})$ for player 2

Mixed Strategies

	L (12.5% vs. 16%)	R
Top (50% vs. 96%)	320, 40	40, 80
Bottom	40, 80	80, 40

- Results deviate from NE
- Player 1 was affected by own payoffs

Ochs (1995)

Player Two

Player One	A	B
A	(1,0)	(0,1)
B	(0,1)	(1,0)

Game One

	A	B
A	(9,0)	(0,1)
B	(0,1)	(1,0)

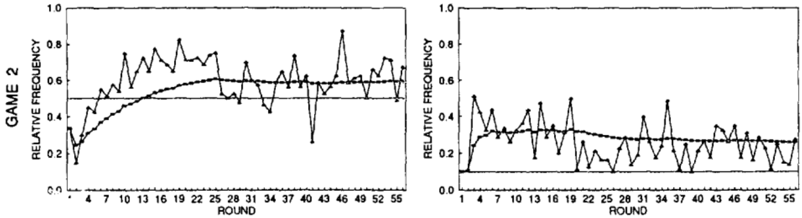
Game Two

	A	B
A	(4,0)	(0,1)
B	(0,1)	(1,0)

Game Three

FIGURE 1

Relative freq. of choosing A over B for Game 2 (16 subjects with "stranger" matching, P1 on the left, P2 on the right):



Observation: Learning doesn't help!

Summary

- If NE is **unique** and in **pure strategies**, behavior converges to it, at least in some games
- If multiple pure strategy NE, can use experiments to see which is selected
  - **Security**, **risk-dominance**, and **principle of insufficient reason** often provide a good explanation
- People don't seem to play mixed strategy NE (except special cases)