1 Normal Form Games

A normal form game is \((I, (A_i)_{i=1,...,n}, (u^i)_{i=1,...,n})\), where \(\forall i A_i\) is an action set, \(A = \times_{i=1}^n A_i\), and \(u_i : A \to \mathbb{R} \ \forall i\).

\[ I = \{1, ..., n\} \] is the set of players.

Assume \(A_i\) is finite for all \(i\).

**Examples:** Coordination, Matching pennies, Prisoner’s dilemma, Battle of the Sexes.

1.1 Dominance

Let \(S^i = \Delta(A^i) = \{(s(a_1^i), ..., s(a_k^i)) : \forall i, s(a_i) \geq 0, \sum_{a_i} s(a_i) = 1\}\).

A mixed extension of a normal form game is \((I, (S^i)_{i=1,...,n}, (u^i)_{i=1,...,n})\), where \(\forall S^i = \Delta(A^i), S = \prod_{i=1}^n S^i\) and \(u^i : S \to \mathbb{R}\) is defined by

\[ u^i(s^1, ..., s^n) = \sum_{a \in A} u^i(a) \prod_{i=1}^n s^i(a^i). \]

We write \(Pr_s(a) = \prod_{i=1}^n s^i(a^i) \in \Delta A\).

Example: Show that in the game below, the player can get a better payoff by mixing T and M than by playing B, no matter what his belief is about what his partner is doing.

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<tr>
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<th>L</th>
<th>R</th>
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<tbody>
<tr>
<td>T</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>M</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We say \(s^i \in S^i\) strictly dominates \(a^i \in A^i\) iff for all \(a^{-i}\)

\[ u^i(s^i, a^{-i}) > u^i(a^i, a^{-i}). \]

Alternatively,

\[ s^i D_2 a^i \Leftrightarrow \forall s^{-i} \in S^{-i} \quad u^i(s^i, s^{-i}) > u_i(a^i, s^{-i}) \]
or

\[ s^i D_3 a^i \iff \forall \mu \in \Delta(A^{-i}) \quad u^i(s^i, \mu) > u_i(a^i, \mu) \]

Exercise: \( s^i D_3 a^i \iff s^i D_2 a^i \iff s^i D_1 a^i \).

Example: Note that T and L are both dominated in the game below.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>-2, -2</td>
<td>-10, -1</td>
</tr>
<tr>
<td>B</td>
<td>-1, -10</td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

This leads to the counter-intuitive prediction of playing (B,R). Of course this doesn’t happen in real life.

Example:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
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</thead>
<tbody>
<tr>
<td>T</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>M</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>B</td>
<td>( x )</td>
<td>( x )</td>
</tr>
</tbody>
</table>

Consider a belief \( p \) for Player 1 that Player 2 chooses L. Note that if \( x < \frac{3}{2} \), B is never a best response. For every belief, Player 1 is better of playing T or M. Dually, \( \exists s^1 \in S^1 \) that dominates B.

If \( x = \frac{3}{2} \), there exists a belief (\( p = 0.5 \)) for which B is a best response. Dually, B is not strictly dominated.

This example suggests that an action is never a best response if and only if it is strictly dominated by a strategy.

**Definition:** An action \( a^i \in A^i \) is never a best response if there is no \( \mu \in \Delta(A^{-i}) \) such that \( u^i(a^i, \mu) \geq u^i(b^i, \mu) \) for all \( b^i \).

**Theorem:** An action \( a^i \in A^i \) is strictly dominated if and only if it is never a best response.

One direction is easy to prove (see your class notes). The proof for the other direction can be found in Osborne and Rubinstein.
1.2 IESDA

We illustrate this with examples:

\[
\begin{array}{cc}
L & R \\
T & 0,-2 & -10,-1 \\
B & -1,-10 & -5,-5 \\
\end{array}
\]

\[
\begin{array}{cc}
L & R \\
T & 3,0 & 0,1 \\
M & 0,0 & 3,1 \\
B & 1,1 & 1,0 \\
\end{array}
\]

Formally, an iterated elimination of strictly dominated actions is a sequence \( A_1, A_2, A_3, ..., A_T \) where:

1. \( \forall k = 1, ..., T \quad A_k = \times_{i=1}^N A^i_k \).
2. \( \forall k, i \quad A^i_{k+1} \subset A^i_k \)
3. \( \forall i \quad A^i_1 = A^i \)
4. Elimination rule: \( \forall k, i \quad a^i \in A^i_k \setminus A^i_{k+1} \) only if \( \exists s^i \in \Delta(A^i_k) \) s.t. \( \forall a^{-i} \in A^{-i}_k \quad u^i(s^i, a^{-i}) > u^i(a^i, a^{-i}) \).

1.3 IEWDA

Definition: \( s^i Wa^i \iff \)

- For all \( a^{-i} \in A^{-i} \) \( u^i(s_i, a^{-i}) \geq u^i(a_i, a^{-i}) \).
- There exists \( b^{-i} \in A^{-i} \) such that \( u^i(s_i, b^{-i}) > u^i(a_i, b^{-i}) \).

Example: Using the game below, show that the set of actions surviving IEWDA is not unique.
1.4 Rationalizability

An action \( a^i \in A^i \) is rationalizable if there exists a vector \((R^1, ..., R^N)\) such that:

1. \( a^i \in R^i \)
2. For all \( j \), \( R^j \subset A^j \)
3. \( \forall j, b^j_i \in R^i \), \( \exists \mu(b^j_i) \in \Delta(A^{-j}) \) (with support \( R^{-j} \)) s.t. \( w^j(b^j_i, \mu) \geq w^j(a^j_i, \mu) \) \( \forall a^j_i \in A^j \).

Example:

\[
\begin{array}{ccc}
L & R \\
T & 1,1 & 0,0 \\
M & 1,1 & 2,1 \\
B & 0,0 & 2,1 \\
\end{array}
\]

\((R^1, R^2) = \{\{M\}, \{R\}\} \).

Example 2:

\[
\begin{array}{ccc}
L & R \\
T & 3,0 & 0,1 \\
M & 0,0 & 3,1 \\
B & 1,1 & 1,0 \\
\end{array}
\]

\((R^1, R^2) = \{\{T\}, \{L\}\} \).

Proposition: If \((R^1, ..., R^N)\) and \((T^1, ..., T^N)\) are rationalizable, then \((R^1 \cup T^1, ..., R^N \cup T^N)\) is rationalizable, as well.

As we discussed before, \( D \leftrightarrow NBR \). \( D \) is related to iterated dominance, while \( NBR \) is related to rationalizability. As we will show below, IESDA and rationaliz-
ability are in some sense equivalent:

\[ IESDA \Leftrightarrow \text{Rationalizability} \]

We will see this from two theorems, the first of which is below:

**Theorem:** Let \( R \) be a set of rationalizable actions. Let \((A_1, ..., A_T)\) be an iterated elimination of strictly dominated actions. Then, \( R^i \subset A^i_T \quad \forall i. \)

**Proof:** The proof proceeds by induction of \( T \), the number of steps in the elimination procedure.

1. (initial step): \( R^i \subset A^i_1 = A^i \) by definition.

2. (inductive step): Assume \( R^i \subset A^i_n \). Let \( a^i \in R^i \). Then,
   \[
   \exists \mu_{a^i} \in \Delta(A^{-i}) \quad \text{(with support } R^{-i}) \quad \text{s.t.} \quad u^i(a^i, \mu_{a^i}) \geq u^i(b^i, \mu_{a^i}) \quad \forall b^i \in A^i.
   \]

Since \( R^i \subset A^i_n \),
   \[
   \exists \mu_{a^i} \in \Delta(A^{-i}_n) \quad \text{s.t.} \quad u^i(a^i, \mu_{a^i}) \geq u^i(b^i, \mu_{a^i}) \quad \forall b^i \in A^i.
   \]

By a theorem we already now, \( D \Rightarrow NBR \). An equivalent of stating this is \( BR \Rightarrow \text{not } D \). Hence,
   \[
   \exists s^i \in \Delta(A^i_n) \quad \text{s.t.} \quad u^i(s^i, a^{-i}) > u^i(a^i, a^{-i}) \quad \forall a^{-i} \in A^{-i}_n.
   \]

I.e., \( a^i \) is not eliminated in the \( n \)th stage of the elimination procedure. Hence, \( a^i \in A^i_{n+1} \), as needed.

**Theorem 2:** Let \( R \) denote the maximal set of rationalizable actions (in terms of set inclusion), and let \((A_1, ..., A_T)\) be a complete elimination of strictly dominated strategies. Then \( A^i_T \subset R^i \) for every \( i \).

**Proof:** It enough to show that \((A^i_1, ..., A^i_N)\) is rationalizable.

Let \( a^i \in A^i_T \). Then \( \exists s^i \in \Delta(A^i_T) \quad \text{s.t.} \quad u^i(s^i, a^{-i}) > u^i(a^i, a^{-i}) \quad \forall a^{-i} \in A^{-i}_T. \)

Note that this implies the same statement with \( s^i \in \Delta(A^i_T) \) replaced by \( s^i \in \Delta(A^i) \).
To see this, let \( s^i \in \Delta(A^i) \) be some strategy that puts a positive probability on an action \( d^i \) that was removed at an earlier step. Since the action was removed, you can find some \( e^i \in A^i_T \) s.t. \( u^i(e^i, a^{-i}) > u^i(d^i, a^{-i}) \) for all \( a^{-i} \in A_T^{-i} \). Transfer the probability placed on \( d^i \) onto \( e^i \). Do the same for all other actions that didn’t survive the elimination procedure to obtain a mixed strategy \( \tilde{s}^i \) with support \( A^i_T \). Note that \( u^i(\tilde{s}^i, a^{-i}) > u^i(s^i, a^{-i}) \) for all \( a^{-i} \in A_T^{-i} \). Since you can’t find a \( \tilde{s}^i \) that dominates \( a^i \) in the last step of the elimination procedure, you will not be able to find a \( s^i \), either.

Next, recall that \( \neg D \Rightarrow BR \). Therefore, \( \exists \mu_{a^i} \in \Delta(A_T^{-i}) \) such that \( u^i(a^i, \mu_{a^i}) \geq u^i(b^i, \mu_{a^i}) \quad \forall b^i \in A^i \).

This shows that \((A^1_T, ..., A^N_T)\) is rationalizable, which is what we needed.

What the previous two theorems show is that that the set of actions that survives \textit{complete} IESDA is unique and equal to the maximal set of rationalizable actions.
Example:

<table>
<thead>
<tr>
<th></th>
<th>b₁</th>
<th>b₂</th>
<th>b₃</th>
<th>b₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>a₁</td>
<td>0.7</td>
<td>2.5</td>
<td>7.0</td>
<td>0.1</td>
</tr>
<tr>
<td>a₂</td>
<td>5.2</td>
<td>3.3</td>
<td>5.2</td>
<td>0.1</td>
</tr>
<tr>
<td>a₃</td>
<td>7.0</td>
<td>2.5</td>
<td>0.7</td>
<td>0.1</td>
</tr>
<tr>
<td>a₄</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
<td>10.0</td>
</tr>
</tbody>
</table>

It’s easy to show that the maximal set of rationalizable actions is \((R^1, R^2) = (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})\) (eliminate \(b_4\) in step 1, and \(a_4\) in step 2).

Example (Cournot Duopoly):

Consider a two player game with two firms \(i = 1, 2\). Each firm faces the demand curve \(p = a - b(q_1 + q_2)\) and per-unit costs of production \(c\). Show that iterated elimination of strictly dominated actions yields a unique outcome in which each firm produces \(a - \frac{c}{3b}\).

1.5 Nash Equilibrium

**Definition (Pure Best Reply):** \(PBR^i(s) = \{a^i \in A^i : u^i(a^i, s^{-i}) \geq u^i(b^i, s^{-i}) \ \forall b^i \in A^i\}\).

Note that this set is nonempty for a finite game.

**Definition (Best Reply):** \(BR^i(s) = \{s^i \in S^i : u^i(s^i, s^{-i}) \geq u^i(b^i, s^{-i}) \ \forall b^i \in A^i\}\).

Note that for every \(s\), \(BR^i(s)\) is closed, convex, nonempty, and equal to the mixed strategies concentrated on \(PBR^i(s)\).

**Example:** Find \(PBR^i(s)\) and \(BR^i(s)\) in the following game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>3.1</td>
<td>0.0</td>
</tr>
<tr>
<td>B</td>
<td>0.0</td>
<td>1.3</td>
</tr>
</tbody>
</table>

**Definition:**

\[ BR(s) = \times_{i=1}^N BR^i(s). \]
Note that $BR : S \to S$ is a closed, convex, and nonempty valued correspondence.

Mathematical Preliminaries for Existence of Nash

The proof of existence of Nash Equilibrium for finite games (what got Nash the Nobel prize) is a simple application of Kakutani’s Fixed Point Theorem. These notes will provide the bare minimum necessary for understanding this theorem. For details, see Mas-Colell, Whinston, and Greene.

**Definition (Correspondence):** Given a set $A \subset \mathbb{R}^N$, a correspondence $f : A \to \mathbb{R}^K$ is a rule that assigns a set $f(x) \subset \mathbb{R}^K$ to $x \in A$.

**Definition (Closed Graph):** Given a set $A \subset \mathbb{R}^N$ and a closed set $Y \subset \mathbb{R}^K$, a correspondence $f : A \to Y$ has a closed graph if for any two sequences $x_n$ and $y_n$ s.t. $x_n \in A \ \forall n, \ y_n \in f(x_n)$ for all $n$, $x_n \to x \in A$, $y_n \to y \in Y$, it is also the case that $y \in f(x)$.

Note that this definition looks very similar to the usual definition of closedness applied to the set $\{(x, y) \in A \times Y : y \in f(x)\}$.

**Definition (Upperhemicontinuity):** Given a set $A \subset \mathbb{R}^N$ and a compact $Y \subset \mathbb{R}^K$, a correspondence $f : A \to Y$ is upperhemicontinuous if it has a closed graph.

Roughly speaking, you can think of upperhemicontinuity as a generalization of the continuity notion for function to correspondences.

**Theorem (Kakutani’s Fixed Point Theorem):** Suppose that $A \subset \mathbb{R}^N$ is a nonempty, compact, convex set and that $f : A \to A$ an upperhemicontinuous correspondence with the property that for each $x \in A$, $f(x) \subset A$ is nonempty and convex. Then $f(\cdot)$ has a fixed point. I.e., there exists an $x \in A$ s.t. $f(x) = x$.

**Definition (Nash Equilibrium):** A Nash Equilibrium of a NFG is a strategy profile $\hat{s}$ s.t. $\hat{s} \in BR(\hat{s})$. 

8
Theorem: The set of Nash Equilibrium strategy profiles is nonempty.

Proof: The set \( S \) is nonempty, compact, and convex. Also, \( BR : S \to S \) is convex and nonempty valued. It remains to show that \( BR(\cdot) \) is upperhemicontinuous. For this, it is sufficient to show that each \( BR^i(\cdot) \) is upperhemicontinuous.

Take a sequence \( s_n \) such that \( s_n \in S \ \forall n \) and a sequence \( t^n_i \) such that \( t^n_i \in BR^i(s_n) \ \forall n, \ s_n \to s \in S, \ t^n_i \to t^i \). We need to show that \( t^i \in BR^i(s) \). This is a simple consequence of continuity of \( u \). In particular, we know that

\[
u^i(t^i_n, s^{-i}_n) \geq u^i(a^i, s^{-i}_n) \ \forall a^i \in A^i, \ \forall n.
\]

Take \( \lim_{n \to \infty} \) of both sides to get:

\[
u^i(t^i_n, s^{-i}_n) \geq u^i(a^i, s^{-i}_n) \ \forall a^i \in A^i.
\]

### 1.6 Constrained Optimization

For what follows, we need to review the theory of constrained optimization. To this end, consider the following problem:

\[
\begin{align*}
\max_{x \in \mathbb{R}^N} & \ f(x) \\
\text{s.t.} & \ g_1(x) = \bar{b}_1 \\
& \vdots \\
& \ g_M(x) = \bar{b}_M \\
& \ h_1(x) \leq \bar{c}_1 \\
& \vdots \\
& \ h_K(x) \leq \bar{c}_K
\end{align*}
\]

Theorem: if \( x^* \) is a local maximum satisfying the constrains then there are multipliers \( \lambda_m \in \mathbb{R} \), one for each equality constraint, and \( \lambda_k \in \mathbb{R}^+ \), one for each inequality constraint, such that:

1. \[
\frac{\partial f(x^*)}{\partial x_n} = \sum_{m}^{M} \lambda_m \frac{\partial g_m(x^*)}{\partial x_n} + \sum_{k}^{K} \lambda_k \frac{\partial h_k(x^*)}{\partial x_n} \text{ for all } n
\]
2. \( \lambda_k (h_k(x^*) - \bar{c}_k) = 0 \) for all \( k \)

In words, the second condition says that \( \lambda_k = 0 \) for each inequality constraint that does not bind.

Sometimes, the conditions above are stated in terms of a Lagrangian function.

Define \( L(x, \lambda) = f(x) + \sum_{m=1}^{M} \lambda_m (\bar{b}_m - g_m(x)) + \sum_{K=1}^{K} \lambda_K (\bar{c}_k - h_k(x)) \).

Then, (1) is equivalent to \( \frac{\partial L(x, \lambda)}{\partial x_n} = 0 \) for all \( n \).

It is common for one of the constraints to take the form of \( x_l \geq 0 \). In this case, the FOC for \( x_l \) changes to \( \frac{\partial L(x, \lambda)}{\partial x_n} \leq 0 \) (show this).

Example: maximize \( f(x, y) = xy \) subject to \( x + y^2 \leq 2 \) and \( x, y \geq 0 \).

1.7 Duality Theorems

A linear program is defined as a triple \((A, b, c)\) s.t. \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n \).

The primal is the following problem

\[ V = \max_{x \geq 0} \{ cx : Ax \leq b \}. \]

The corresponding dual is

\[ W = \min_{y \geq 0} \{ yb : yA \geq c \}. \]

The first of the duality theorems shows that the two solutions are related:

**Theorem (Weak Duality):** \( W \geq V \).

**Proof.** If the primal is infeasible, then \( V = -\infty \) by convention. Similarly, \( W = \infty \) if the dual is infeasible. Hence, if either problem is infeasible, then \( W \geq V \).

Consider the case where both problems are feasible. Then, for all feasible \( x \) and \( y \),

\[ cx \leq yb. \]
Taking sup w.r.t. feasible $x$, $V \leq yb$. Taking inf w.r.t. feasible $y$, $V \leq W$. This finishes the proof.

It turns out that in many cases we can make a statement that’s stronger. This statement is given below with omission of the proof:

If the primal and the dual are feasible, then both have optimal solutions, and $V = W$.

Armed with this result, we can interpret the solution to the dual as Lagrange multipliers for the primal, and vice versa.

**Theorem (Complementary Slackness):**

The feasible pair $(x^*, y^*)$ is optimal

$\Leftrightarrow$

(i) $x_j^* > 0 \Rightarrow y^* a_j = c_j$ and $y^* a_j > c_j \Rightarrow x_j^* = 0$;

(ii) $y_i^* > 0 \Rightarrow a_i x^* = b_i$ and $a_i x^* > b_i \Rightarrow y_i^* = 0$.

**Proof.** $(\Rightarrow)$ Conditions (i) and (ii) above can be rewritten as

\[
(y^* A - c)x^* = 0, \quad (1.1)
\]

\[
y^*(Ax^* - b) = 0. \quad (1.2)
\]

If $(x^*, y^*)$ is optimal, then $cx^* = y^* b$ by the same logic as in the proof of weak duality. By feasibility, $y^* A \geq c \Rightarrow y^* Ax^* \geq cx^*$. Similarly, $Ax^* \leq b \Rightarrow y^* Ax^* \leq y^* b$. Hence, $y^* Ax^* = y^* b = cx^*$.

$(\Leftarrow)$ If 1.1 and 1.2 hold for some feasible $(x^*, y^*)$, then $cx^* = y^* b$. By weak duality, it follows that $(x^*, y^*)$ are optimal solutions to the primal and the dual. This finishes the proof.

It follows that the Lagrangian for

\[
V = \max_{x \geq 0} \{ cx : Ax \leq b \}
\]

is

\[
L(x, y) = cx + y(b - Ax),
\]
and the Lagrangian for

\[ W = \min_{y \geq 0} yb : yA \geq c \]

is

\[ M(x, y) = -yb + (yA - c)x. \]

Notice that \( L(x, y) = -M(x, y) \), i.e. the two problems are different sides of the same coin!

We can think of \( x \) and \( y \) as controlled by two competing players. If the \( x \) player chooses something that violates \( Ax \leq b \), the \( y \) player makes \( L(x, y) \) go to \(-\infty\), same for \( M(x, y) \). This intuition invites us to think about zero-sum games.

### 1.8 Application: Zero-sum Games

We are now equipped to think about zero-sum games in the same terms as Von-Neumann and Morgenstern in their famous book. Let’s start with an example.

**Example 1.4 (Rock Paper Scissors)**

<table>
<thead>
<tr>
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<th>R</th>
<th>S</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>R</strong></td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td><strong>S</strong></td>
<td>-1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td><strong>P</strong></td>
<td>1</td>
<td>-1</td>
<td>0</td>
</tr>
</tbody>
</table>

Row chooses \((y_1, y_2, y_3) \in \Delta^3\), column chooses \((x_1, x_2, x_3) \in \Delta^3\).

Consider row’s paranoid problem, i.e. a problem in which row assumes column is out to get him.

\[
V = \min_{x \geq 0, R} R \quad s.t. \quad x_2 - x_3 \leq R \quad /y_1, \\
- x_1 + x_3 \leq R \quad /y_2, \\
x_1 - x_2 \leq R \quad /y_3, \\
x_1 + x_2 + x_3 = 1 \quad /C.
\]
The dual of this is

\[ W = \max_{y \geq 0, C} C \quad s.t. \quad C \leq y_3 - y_2, \]

\[ C \leq y_1 - y_3, \]

\[ C \leq y_2 - y_1, \]

\[ y_1 + y_2 + y_3 = 1 \]

i.e. a paranoid problem for the column player! (Constraints reflect the paranoia.) By strong duality, both problems have optimal solutions and \( V = W \).

This result is general. Assume a 2 player game with payoff matrix \( A \in \mathbb{R}^{m \times n} \) (the row player has \( m \) pure strategies, and the column player has \( n \) pure strategies).

Define **row’s paranoid problem** as

\[ \min_{x \geq 0, R} R \quad s.t. \quad \sum_{j=1}^{n} a_{ij} x_j \leq R \quad \forall i \in \{1, \ldots, m\}, \]

\[ \sum_{j=1}^{n} x_j = 1, \]

and **column’s paranoid problem** as

\[ \max_{y \geq 0, C} C \quad s.t. \quad \sum_{i=1}^{m} y_i a_{ij} \geq C \quad \forall j \in \{1, \ldots, n\}, \]

\[ \sum_{i=1}^{m} y_i = 1. \]

\((x^*, y^*)\) is a **Nash Equilibrium** if

\[ y^* Ax^* \leq y^* Ax^* \leq y^* Ax \quad \forall (x, y) \in \Delta^m \times \Delta^n. \]

**Theorem:** A Nash Equilibrium exists in a zero-sum game.

**Proof.** Using complementary slackness,

\[ y^* Ax^* = R \geq a_i x^* \quad \forall i \in \{1, \ldots, m\}. \]

Hence, \( y^* Ax^* \geq y^* Ax^* \quad \forall y \in \Delta^m. \)

Similarly, \( y^* Ax \geq y^* Ax^* \). This finishes the proof.

A direct consequence of duality in zero sum games is the following theorem:
Theorem (Minimax Theorem):

\[
\min_{x \in \Delta^n} \max_{i} \{ a_i x \} = \min_{x \in \Delta^n} \max_{y \in \Delta^m} \{ yAx \} = \max_{y \in \Delta^m} \min_{x \in \Delta^n} \{ yAx \} = \max_{y \in \Delta^m} \min_{j} \{ y a_j \}.
\]

In words, the theorem tells us that it doesn’t matter whether row or column goes first.

1.9 Correlated Equilibrium

The first step is to recognize that the set of probabilities induced on \( A \) by mixed strategies is not equal to \( \Delta(A) \). Let \( Pr_s(a) = \prod_{i=1}^{N} s^i(a^i) \). The claim is that

\[
\{Pr_s(a)\} \subsetneq \Delta(A).
\]

**Example:** Consider a \( \mu(a) \) given by:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>B</td>
<td>0</td>
<td>1/2</td>
</tr>
</tbody>
</table>

The set \( \{Pr_s(a)\} \) is:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>( s^1(T)s^2(L) )</td>
<td>( s^1(T)s^2(R) )</td>
</tr>
<tr>
<td>B</td>
<td>( s^1(B)s^2(L) )</td>
<td>( s^1(B)s^2(R) )</td>
</tr>
</tbody>
</table>

Clearly, \( \mu(a) \notin \{Pr_s(a)\} \).

A **correlated equilibrium** is a probability distribution \( \mu(a) \) such that if players are given recommendations based on the distribution, any player \( i \) finds it in his best interests to play \( a^i \), assuming all other players do the same.

**Example:**

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>3,1</td>
<td>0,0</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>1,3</td>
</tr>
</tbody>
</table>

Any \( \mu(a) \in \Delta(A) \) is vector \( p \) s.t. \( \sum p_{ij} = 1 \), where ant \( p_{ij} \geq 0 \) is the probability assigned to the action \( a_{ij} \) in the payoff matrix:
Assume that the players are given recommendations based on $\mu$. Then,

- $\mu(T) = \left( \frac{p_{11}}{p_{11}+p_{12}}, \frac{p_{12}}{p_{11}+p_{12}} \right)$,
- $\mu(B) = \left( \frac{p_{21}}{p_{21}+p_{22}}, \frac{p_{22}}{p_{21}+p_{22}} \right)$,
- $\mu(L) = \left( \frac{p_{11}}{p_{11}+p_{21}}, \frac{p_{21}}{p_{11}+p_{21}} \right)$,
- $\mu(R) = \left( \frac{p_{12}}{p_{12}+p_{22}}, \frac{p_{22}}{p_{12}+p_{22}} \right)$.

It follows that

- Player 1 follows the recommendation $T$ if $3 \cdot p_{11} \geq 1 \cdot p_{12}$,
- Player 1 follows the recommendation $B$ if $1 \cdot p_{22} \geq 3 \cdot p_{21}$,
- Player 2 follows the recommendation $L$ if $1 \cdot p_{11} \geq 3 \cdot p_{21}$,
- Player 2 follows the recommendation $R$ if $3 \cdot p_{22} \geq 1 \cdot p_{12}$.

The set of correlated equilibria is the set of $p$’s satisfying these inequalities. Note that any $p$ derived from a Nash equilibrium works! Thus, the inequalities are satisfied by $(1, 0, 0, 0)$, $(0, 0, 0, 1)$, and $(\frac{3}{16}, \frac{9}{16}, \frac{1}{16}, \frac{3}{4})$. The question is if the set of correlated equilibria is bigger. And indeed it is. Clearly, any convex combination of the three correlated equilibria above is itself a correlated equilibrium.

Let’s make this a bit more formal.

**Definition (Correlated Equilibrium):** $\mu \in \Delta(A)$ is a correlated equilibrium if $\forall i, a^i \in A^i, b^i \in A^i$,

$$\sum_{a^{-i}} u^i(a^i, a^{-i})\mu(a^{-i}|a^i) \geq \sum_{a^{-i}} u^i(b^i, a^{-i})\mu(a^{-i}|a^i).$$

**Theorem:** The set of correlated equilibria is non-empty, closed, and convex.
**Proof:** It is non-empty because any $Pr_s(a)$ derived from a Nash Equilibrium leads to a correlated equilibrium (check it). It is closed and convex because it is a system of linear inequalities. This finishes the proof.

**Example (Game of Chicken):**

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>6,6</td>
<td>2,7</td>
</tr>
<tr>
<td>B</td>
<td>7,2</td>
<td>0,0</td>
</tr>
</tbody>
</table>

This game has three Nash equilibria, with payoffs (2,7), (7,2), and ($4\frac{2}{3}, 4\frac{2}{3}$). The $\mu$ below gives a correlated equilibrium with payoffs (5,5), which is outside the convex hull of Nash equilibrium payoffs.

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>$\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>B</td>
<td>$\frac{1}{3}$</td>
<td>0</td>
</tr>
</tbody>
</table>

**Example (Correlated Equilibrium in an Infinite Game)**

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,2,3 $+1/\ell$</td>
<td>0,0,8</td>
</tr>
<tr>
<td>D</td>
<td>0,0,0</td>
<td>2,2,0</td>
</tr>
</tbody>
</table>

$z < 0$

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,2,2</td>
<td>0,0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0,0</td>
<td>2,2,2</td>
</tr>
</tbody>
</table>

$z = 0$

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,2,0</td>
<td>0,0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0,8</td>
<td>2,2,3 $-1/\ell$</td>
</tr>
</tbody>
</table>

$z > 0$

Here, $z \in \mathbb{Z}$, so player 3 has infinitely many actions.

This game has no pure or mixed strategy Nash Equilibria. To see this, notice that regardless of what player 3 is doing, players 1 and 2 will be playing the following game between themselves:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>2,2</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>2,2</td>
</tr>
</tbody>
</table>

The Nash Equilibria of this game are $\{\{1,0\}, \{1,0\}\}$, $\{\{1/2,1/2\}, \{1/2,1/2\}\}$, $\{\{0,1\}, \{0,1\}\}$.

If (U,L) is played, 3 wants do decrease $z$ to $-\infty$. If (D,R) is played, 3 wants to increase $z$ to $\infty$. Similarly, in the mixed case Player 3 has no best response.
On the other hand, a correlated equilibrium exists. To see this, let \( \mu(U, L, z = 0) = \mu(D, R, z = 0) = 1/2 \).

### 1.10 Correlated Equilibrium as an Expression of Bayesian Rationality

Another important interpretation of correlated equilibria was provided by Aumann (1987). Assume we are exogenously given the following informational model of the world:

(i) A finite set \( \Omega \), with generic element \( \omega \),
(ii) A probability measure \( \rho \) on \( \Omega \),
(iii) \( \forall i \), a partition \( P^i \) of \( \Omega \).

Note: Each \( \omega \in \Omega \) should be thought of as a specification of all parameters of interest to any player \( i \) (in particular, a vector of moves of all other players).

Denote by \( \sigma^i(\omega) \) the action chosen by player \( i \) at the state \( \omega \). The information structure is such that \( \sigma^i(\omega_1) = \sigma^i(\omega_2) \) \( \forall \omega_1, \omega_2 \in P \in \mathcal{P}^i \), i.e. the player cannot distinguish between elements of \( P \).

We say that a player \( i \) is **Bayes rational** at \( \omega \) if

\[
E(u^i(\sigma)|\mathcal{P}^i)(\omega) \geq E(u^i(a^i, \sigma_{-i})|\mathcal{P}^i)(\omega) \quad \forall a^i \in A^i,
\]

where for any \( \omega \in P \in \mathcal{P}^i \), \( E(x|\mathcal{P}^i)(\omega) = E(x|P) \).

**Aumann’s Theorem:**

(1) Given a correlated equilibrium \( \mu \), we can find \( (\Omega, \rho, \mathcal{P}, \sigma) \) such that each player is Bayes rational.

(2) If each player is Bayes rational at each state of the world, then \( \mu \in \Delta(A) \) induced by \( \sigma \) is a correlated equilibrium distribution. That is, if \( \mu(a) = \rho(\{w \mid \sigma^i(\omega) = a^i \quad \forall i\}) \quad \forall a \in A \), then \( \mu \) is a correlated equilibrium.
Proof. (1) Let
\[ \Omega = A, \quad \mathcal{P}^i = \{\{a^i\} \times A^{-i} | a^i \in A^i\}, \quad \sigma^i(\{a^i\} \times A^{-i}) = a^i. \]

Then,
\[ \rho(a^{-i}|a^i) = \frac{\mu(a^i, a^{-i})}{\sum_{\tilde{a}^{-i}} \mu(a^i, \tilde{a}^{-i})}. \]

Hence, \( \forall i, \forall a^i \in A^i \text{ s.t. } \mu(a^i) > 0, \forall b^i \in A^i, \)
\[ \sum_{a^{-i}} u^i(a)\rho(a^{-i}|a^i) \geq \sum_{a^{-i}} u^i(b^i, a^{-i})\rho(a^{-i}|a^i). \]
\[ \Rightarrow \sum_{\omega \in P} u^i(\sigma(\omega))\rho(w|P) \geq \sum_{\omega \in P} u^i(b^i, \sigma^{-i}(\omega))\rho(w|P), \quad P = a^i \times A^{-i} \in \mathcal{P}^i \]
\[ \Rightarrow E(u^i(\sigma)|\mathcal{P}^i)(\omega) \geq E(u^i(\sigma^{-i}, a^i)|\mathcal{P}^i)(\omega) \quad \forall b^i \in A^i, \]

Where \( \omega \in P. \) Since \( \mu(a^i) = 0 \Rightarrow \rho(a^{-i}|a^i) = 0, \) the above extends to all \( \omega. \) Hence, each player is Bayes rational.

(2) \( \forall i, \) define \( \mathcal{H}^i(a^i) = \{\omega : \sigma^i(\omega) = a^i\}. \) This could be a single information set of player \( i \) or a union of many. In any case, it is a coarser partition than \( \mathcal{P}^i. \)

Every player is Bayes rational, so
\[ E(u^i(\sigma)|\mathcal{P}^i)(\omega) \geq E(u^i(b^i, \sigma^{-i}(\omega))|\mathcal{P}^i)(\omega) \quad \forall i, b^i \in A^i, w \in \Omega \]
\[ \Leftrightarrow \sum_{\omega \in P} u^i(\sigma(\omega))\rho(\omega) \geq \sum_{\omega \in P} u^i(b^i, \sigma^{-i}(\omega))\rho(\omega) \quad \forall i, b^i \in A^i, P \in \mathcal{P}^i \]
\[ \Rightarrow \sum_{\omega \in \mathcal{H}^i(a^i)} u^i(\sigma(\omega))\rho(\omega) \geq \sum_{\omega \in \mathcal{H}^i(a^i)} u^i(b^i, \sigma^{-i}(\omega))\rho(\omega) \quad \forall i, a^i, b^i \in A^i \]
\[ \Rightarrow \sum_{\omega \in \mathcal{H}^i(a^i, \sigma^{-i}(\omega))\rho(\omega) \geq \sum_{\omega \in \mathcal{H}^i(a^i, \sigma^{-i}(\omega))} u^i(b^i, \sigma^{-i}(\omega))\rho(\omega) \quad \forall i, a^i, b^i \in A^i \]
\[ \Rightarrow \sum_{a^{-i}} u^i(a^i, a^{-i})\mu(a) \geq \sum_{a^{-i}} u^i(b^i, a^{-i})\mu(a) \quad \forall i, a^i, b^i \in A^i. \]

Example:

Consider the following three-player normal form game.
The pure strategy Nash Equilibria of this game are \((D, L, A)\), \((U, R, A)\), \((D, L, C)\), \((U, R, C)\). Notice that \(B\) is never played in a Nash Equilibrium. There is, however, a way to make player 3 (who controls the matrices) play \(B\) with probability 1 in correlated equilibrium.

To this end, let \(\mu(U, L, B) = \mu(D, R, B) = \frac{1}{2}\). It’s trivial to see that the definition of correlated equilibrium is satisfied.

Using Aumann’s theorem, each player is Bayes rational in a game augmented by the following information structure:

\[
\Omega = \text{action profiles}, \quad \rho = \mu, \quad \mathcal{P}_3 = \{\{(A, U, L), (A, U, R), (A, D, L), (A, D, R)\}, \{(B, U, L), (B, U, R), (B, D, L), (B, D, R)\}, \{(C, U, L), (C, U, R), (C, D, L), (C, D, R)\}\}, \text{etc.}
\]

### 1.11 Perfect Equilibrium

The idea of perfect equilibrium (PE) is motivated by the possibility that nobody’s perfect (people make mistakes). As motivation, consider the following game:

\[
\begin{array}{ccc}
\text{L} & \text{R} \\
\text{U} & 0,0,3 & 0,0,0 \\
\text{D} & 1,0,0 & 0,0,0 \\
\end{array}
\begin{array}{ccc}
\text{L} & \text{R} \\
\text{U} & 2,2,2 & 0,0,0 \\
\text{D} & 0,0,0 & 2,2,2 \\
\end{array}
\begin{array}{ccc}
\text{L} & \text{R} \\
\text{U} & 0,0,0 & 0,0,0 \\
\text{D} & 0,1,0 & 0,0,3 \\
\end{array}
\]

This game has two NE in pure strategies, \((T, L)\) and \((B, R)\), and no other mixed or pure strategy equilibria. However, the equilibrium \((B, R)\) is not robust to small mistakes. To see this, let \(\epsilon\) be the smallest probability of any action being chosen. Player 1’s payoff from any mixed strategy profile \(((p, 1 - p), (q, 1 - q))\) is \(pq\). When the possibility of mistakes is introduced, \(q \geq \epsilon\). Thus, Player 1 chooses the maximum \(p\) he can, which is \(1 - \epsilon\). Likewise, Player 2 will put probability \((1 - \epsilon)\) on L against any mixed strategy of Player 1. Thus, the only equilibrium of the perturbed game is \(((1 - \epsilon, \epsilon), (1 - \epsilon, \epsilon))\). This equilibrium goes to \((T, L) = ((1, 0), (1, 0))\) as the probability
of making a mistake goes to 0. (T,L) is called a perfect equilibrium.

In general, given any \( \epsilon = ((\epsilon_1^1, \ldots, \epsilon_{A_1}^1), \ldots, (\epsilon_1^N, \ldots, \epsilon_{A_N}^N)) \), define

\[
S^i_\epsilon = \{ s^i \in S^i : s^i(a^i_j) \geq \epsilon^i_j \ \forall j \in \{1, 2, \ldots, \# A^i\}\}.
\]

**Definition:** Given any mixed extension of a normal form game \( G \), the perturbed game \( G^\epsilon \) consists of the same players and the same utility functions, but with \( S^i \) replaced by \( S^i_\epsilon \) for every \( i \).

**Definition:** The set of Nash Equilibria of the perturbed game is written as \( NE(G^\epsilon) \) and defined as all \( s \in S^i_\epsilon \) such that

\[
u^i(s^i, s^{-i}) \geq u^i(t^i, s^{-i}) \ \forall i, \forall t^i \in S^i_\epsilon.
\]

**Definition:** A mixed strategy \( s \in S \) is called a perfect equilibrium if there exist sequences \( \{\epsilon(n)\} \) and \( \{s(n)\} \) such that the following conditions are satisfied:

1. \( \epsilon^j_i(n) > 0 \ \forall i \in \{1, \ldots, N\}, j \in \{1, \ldots, \# A^i\}, n \in \{1, 2, 3, 4, \ldots\} \).
2. \( \sum_j \epsilon^j_i(n) < 1 \ \forall i, n. \)
3. \( \lim_{n \to \infty} \epsilon^j_i(n) = 0 \ \forall i, j. \)
4. \( s(n) \in NE(G^\epsilon(n)) \ \forall n. \)
5. \( \lim_{n \to \infty} s(n) = s. \)

**Theorem:**

1. Given any \( \epsilon = ((\epsilon_1^1, \ldots, \epsilon_{A_1}^1), \ldots, (\epsilon_1^N, \ldots, \epsilon_{A_N}^N)) \), the set \( NE(G^\epsilon) \) is nonempty.
2. The set of perfect equilibria is nonempty.
3. Any perfect equilibrium is a Nash equilibrium.

**Proof:**

1. Homework.
2. Let $\epsilon(n)$ be any sequence s.t. conditions 1-3 in the definition of perfect equilibrium are satisfied, e.g. $\epsilon(n) = \frac{1}{n} \forall n$ works for large enough $n$. By part 1 of the theorem, $\exists s(n) \in NE(G^\epsilon(n)) \forall n$. $s(n)$ is a sequence in $S$, a compact set, and hence has a convergent subsequence $s(m(n)) \to s \in S$. The limit of the sequence is a perfect equilibrium by definition.

3. Since $s$ is a perfect equilibrium, we know that there exists $s(n)$ s.t.

$$u^i(s^i(n), s^{-i}(n)) \geq u^i(r^i, s^{-i}(n)) \quad \forall i \forall r^i \in S^i_{\epsilon(n)}.$$  

Let $t^i$ be any action in $S^i$. Since $\epsilon(n) \to 0$, we can find a sequence $t^i(n) \to t^i$. Then,

$$u^i(s^i(n), s^{-i}(n)) \geq u^i(t^i(n), s^{-i}(n)) \quad \forall i \forall r^i \in S^i_{\epsilon(n)}.$$  

Taking $\lim_{n \to \infty}$ of both sides of the inequality we get

$$u^i(s^i, s^{-i}) \geq u^i(t^i, s^{-i}) \quad \forall i \forall s^i \in S^i.$$  

This shows that $s$ is a Nash Equilibrium.

1.12 Evolutionarily Stable Strategies

This is another refinement which applies evolutionary reasoning to game theoretic concepts. In particular, we now interpret players as animals or plants with strategies chosen by genetics, and players’ utilities as fitness. Fitness can be thought of as the animal’s survival potential.

The idea of being evolutionarily stable is the following. With some small probability $\epsilon$, nature may introduce a mutant into a population of healthy agents playing some strategy $b^*$. We say that $b^*$ is evolutionarily stable if it has greater survival potential than the mutant. In this case, the mutant will die out.

In particular, consider a population composed of $1 - \epsilon$ players with strategy $b^*$ and $\epsilon$ mutants with strategy $b$. $b^*$ has greater fitness than $b$ if

$$(1 - \epsilon)u(b^*, b^*) + \epsilon u(b^*, b) > (1 - \epsilon)u(b, b^*) + \epsilon u(b, b).$$  

This condition is satisfied for some small $\epsilon$ if either of the following hold:

1. $u(b^*, b^*) > u(b, b^*)$;
2. \( u(b^*, b^*) = u(b, b^*) \) and \( u(b^*, b) > u(b, b) \).

Motivated by this, we introduce the following definition:

**Definition (ESS):** Let \( G = (\{1, 2\}, \{B, B\}, \{u^i\}_{i=1,2}) \) be a symmetric game. \( b^* \in B \) is an **evolutionarily stable strategy (ESS)** if \( (b^*, b^*) \in NE(G) \) and \( u(b^*, b) > u(b, b) \) for all \( b \in BR(b^*) \).

Note: ESS is a refinement on NE, since it gives a more restrictive prediction.

Note 2: Any NE \( b^* \) with the property that no strategy other than \( b^* \) is a best response to \( b^* \) is a ESS. Nash equilibria with this property are called **strict** Nash equilibria.

**Example:** A non-strict NE may not be an ESS. Consider the 2x2 symmetric game where \( u(a, b) = 1 \) \( \forall a, b \). Everything is a NE here, and nothing is ESS.

**Example 2:** Consider the following game, with \( 0 < \gamma < 1 \):

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>( \gamma, \gamma )</td>
<td>1,-1</td>
<td>-1, 1</td>
</tr>
<tr>
<td>B</td>
<td>-1, 1</td>
<td>( \gamma, \gamma )</td>
<td>1, -1</td>
</tr>
<tr>
<td>C</td>
<td>1,-1</td>
<td>-1, 1</td>
<td>( \gamma, \gamma )</td>
</tr>
</tbody>
</table>

\( (1/3,1/3,1/3) \) is the unique mixed strategy NE of this game. However, it is not evolutionary stable.

**Example 4:** Consider the following game:

\[
\begin{array}{ccc}
0,0 & 2,2 \\
2,2 & 0,0
\end{array}
\]

The unique mixed NE of this game is \( ((1,2),(1,2)) \), and it is ESS.

**Example 5 (Hawk/Dove):**

\[
\begin{array}{ccc}
1/2, 1/2 & 0,1 \\
1,0 & 1/2(1-c), 1/2(1-c)
\end{array}
\]
2 Extensive Form Games

An extensive form game has the following basic ingredients: set of players $I$ (possibly including nature), set of nodes $X$, set of final nodes $Z$, original node $o$, and set of utility functions $u^i : Z \to \mathbb{R}$. Any description of an EFG must also specify:

- How the nodes are ordered;
- How the nodes are allocated among players, including nature;
- How nature determines its moves;
- What information the players have;
- What moves the player can make at every node.

Ordering of nodes

Formally, we require $\succsim$ a partial order over the set of nodes ($x \succsim y$ means $x$ comes before $y$) with the following properties:

1. $\succsim$ is reflexive, antisymmetric, transitive;
2. $\forall x \ 0 \succsim x$;
3. $\forall x \in X, \ p(x)=\text{the set of predecessors of } x=\{y : y \succsim x\}$ is linearly ordered by $\succsim$.

Players’ partition

A players’ partition is partition $(P^1, P^2, ..., P^N)$ of the set of non-terminal nodes:

1. $\forall i \neq j, \ P^i \cap P^j = \emptyset$;
2. $\bigcup_i P^i = X \setminus \{o, Z\}$. 
Nature’s move, if nature is a player

The probability of nature’s move is a probability distribution $p_o \in \Delta(S(o))$, where $S(o)$ is the set of immediate successors of the initial node.

Information partition

For every player $i$, the information partition $U^i = \{u, v, w, ...\}$ is a partition of $P^i$. That is,

1. $u \cup v \cup w \cup ... = P^i$;
2. $u \neq v \Rightarrow u \cap v = \emptyset$.

Players’ choices

Let $S(x)$ the set of immediate successors of node $x$. For any information set $u$, let $\bigcup_{x \in u} S(x)$ to be the set of immediate successors of $u$. For all $i, u \in U^i$, $C_u$ is the set of choices $i$ can make at $u$. This is a partition of $\bigcup_{x \in u} S(x)$ such that each set contains one and only one of the elements of $S(x)$ for every $x \in u$.

Additional properties

We might want to impose some additional intuitive properties on extensive form games. In particular, we might want to rule out situations where a player doesn’t remember if he moved or not, and situations where the player has different numbers of actions at different nodes in the same information set.

Formally, we might assume that the game is linear. This means that $\forall i, \forall u \in U^i, \forall z \in Z, \#\{p(z) \cap u\} \leq 1$. In words, the path of predecessors of any terminal node intersects any information set at most once. This rules out situations where players forget whether or not they moved.

We might also assume that $\forall i, \forall u \in U^i, \forall x_1, x_2 \in u, \#S(x_1) = \#S(x_2)$. This rules out situations where different nodes in the same information set have different
numbers of actions. This would be nonintuitive since we expect that a player would be able to distinguish between such nodes.

2.1 Strategies in Extensive Form Games

Definition (Pure Strategies): A pure strategy specifies a choice at every information set. Formally, the set of pure strategies \( A^i = \times_{u \in U^i} C^i_u = \{(a^i_u)_{u \in U^1} : a^i_u \in C^i_u \ \forall u\} \).

Definition (Mixed Strategies): A mixed strategy is a probability distribution on \( A^i \). Formally, \( S^i = \Delta(A^i) \).

Example:

\[ X = \{x_1, x_2, z_1, z_2, z_3\}, \ Z = \{z_1, z_2, z_3\}, \ I = \{1\}, \ u^1(z_i) = i. \]

The order over the set of nodes is the following: \( x_1 \succ z_1, \ x_1 \succ x_2, \ x_2 \succ z_2, \ x_2 \succ z_3. \) Player 1 moves at both \( x_1 \) and \( x_2 \): \( P^1 = \{x_1, x_2\}. \) Player 1 cannot distinguish between being at \( x_1 \) and \( x_2 \): \( U^1 = \{P^1\}. \) The choices of Player 1 are given by the following: \( C_{P^1} = \{\{z_1, z_2\}, \{x_2, z_3\}\}. \)

Questions: Draw this extensive form game. Is the game linear? What is the set of probabilities on \( Z \) that you can achieve with Player 1’s pure strategies? What is the set of probabilities on \( Z \) that you can achieve with Player 1’s mixed strategies?

2.2 Normal Representation

Every extensive form game has a normal form representation, with the same set of players \( I \) and \( A^i = \times_{u \in U^i} C^i_u \), as defined above. Every action profile leads to a terminal node, and \( u^i(z) \) at that node gives player \( i \)’s utility. If nature is one of the players, then for every pure strategy profile \( (a^1, ..., a^N) \) of the players, there is a unique probability on the final nodes given by \( P^i_{p, a} \in \Delta(Z) \) and hence an expected utility vector. Thus, in general, the NFG representation of an EFG has:

- \( I = \{1, ..., N\}; \)
- For every \( i \), \( A^i \) the set of pure strategies;
• \( u^j(a) = E_{Pr_{pg}} u^j(\cdot) \).

2.3 NE of EFG

**Definition (NE of EFG):** The NE of the EFG is the NE of the mixed extension of the associated NFG.

Note: We say that two strategies \( d^i \) and \( e^i \) are payoff equivalent if \( u^j(a^{-i}, d^i) = u^j(a^{-i}, e^i) \forall a^{-i} \in A^{-i}, \forall j \in I \). I.e., no player cares if \( i \) uses \( c^i \) or \( d^i \). We can replace each set of payoff-equivalent strategies with a single strategy and relabel. The result is called the purely reduced normal representation. The definition of NE above uses the NFG representation of an EFG, and not the purely reduced normal representation. The pure strategies in the NFG representation of EFG must be exactly the pure strategies of the EFG.

2.4 Behavioral Strategies

**Definition (Behavioral Strategies):** A behavioral strategy specifies a probability distribution for every information set for every player that the player uses to randomize. Formally, the set of behavioral strategies is of player \( i \) is:

\[
B^i = \{(b^u)_{u \in U^i} : b^i \in \Delta(C_u) \ \forall u \in U^i\}.
\]

**Definition (General Strategies):** The set of general strategies for player \( i \) is the set of probability distributions on \( B^i \).

**Example:** Continue with the game in the previous example. What is the set of probabilities on \( Z \) that you can achieve with Player 1’s behavioral strategies?

**Definition:** Two strategies of player \( i \) are equivalent if they induce the same probability distribution on \( Z \) for any general strategy of the other players.

Comparing the probability distributions on \( Z \) induced by mixed and behavioral strategies in the previous two examples shows that in general a behavioral strategy is not equivalent to a mixed strategy. Nor is a mixed strategy equivalent to a behavioral strategy.
**Example:** At node $x_1$ (the only node in the information set $u$), Player 1 chooses between $L_1$ and $R_1$. The immediate successors of $x_1$ are $x_2$ and $x_3$. $L_1$ leads to node $x_2$ and $R_1$ leads to node $x_3$. $x_2$ and $x_3$ are contained in the same information set $v$, where Player 1 chooses between $L_2$ and $R_2$. The immediate successors of $x_2$ are $z_1$ and $z_2$. The immediate successors of $x_3$ are $z_3$ and $z_4$. $L_2$ selects $z_1$ and $z_3$, and $R_2$ selects $z_2$ and $z_4$.

- $A^1 = \{L_1, R_1\} \times \{L_2, R_2\};$
- $S^1 = \{(s^1(L_1, L_2), s^1(L_1, R_2), s^1(R_1, L_2), s^1(R_1, R_2))\}$ s.t. each element of the vector is possible and all of them add up to 1;
- $B^1 = \{(p(1-p)(q, 1-q))\}$, where $p$ is the probability of choosing $L_1$ at $u$ and $q$ is the probability of choosing $L_2$ at $v$.

Note that *any* probability on final nodes can be achieved with mixed strategies.

The set of probability distributions on the final nodes achievable by behavioral strategies is \{$(pq, p(1-q), (1-p)q, (1-p)(1-q))\}$ with $p, q \in [0, 1]$. Note that this set is strictly smaller than the set of all probability distributions on the final nodes.

### 2.5 Kuhn’s Theorem

The examples above suggest that mixed and behavioral strategies are not equivalent in general. They are, however, equivalent in games with the property of perfect recall.

**Definition (Perfect Recall):** A game satisfies perfect recall if there exists no player $i$ with information sets $u$ and $v$ and a choice $c \in C_u$ such that $x_1, x_2 \in v$, $x_1$ follows $c$ but $x_2$ does not follow $c$.

**Proposition:** Perfect recall implies that the game is linear.

**Theorem (Kuhn’s Theorem):** In any EFG where player $i$ has perfect recall, for any mixed strategy of player $i$ there is an equivalent behavioral strategy and vice versa.

**Proof:** See Osborne in Rubinstein.
Given a mixed strategy $s^i$, we define the behavioral strategy as follows:

$$b^i_{u \in U^i}(c) = \frac{\text{Probability of reaching } u \text{ and making choice } c \text{ with } s^i}{\text{Probability of reaching } u \text{ with } s^i}.$$ 

Given a behavioral strategy $b^i$, we define the mixed strategy as follows:

$$s^i(a) = \Pi_{u \in U^i} b^i_{u}(c_u(a)),$$

where $c_u(a)$ is the choice specified by $a$ at information set $u$.

**Example:** At node $x_1$ (the only node in the information set $u$), Player 1 chooses between $L_1$ and $R_1$. The immediate successors of $x_1$ are $z_1$ and $x_2$. $L_1$ leads to node $z_1$ and $R_1$ leads to node $x_2$, which is the sole element of the information set $v$. At $x_2$, Player 1 chooses between $L_2$ and $R_2$. The immediate successors of $x_2$ are $z_2$ and $z_3$. $L_2$ selects $z_2$, and $R_2$ selects $z_3$.

- $S^1 = \{(s^1(L_1, L_2), s^1(L_1, R_2), s^1(R_1, L_2), s^1(R_1, R_2))\}$ s.t. each element of the vector is positive and all of them add up to 1;
- $B^1 = \{(p, 1 - p)(q, 1 - q))\}$, where $p$ is the probability of choosing $L_1$ at $u$ and $q$ is the probability of choosing $L_2$ at $v$.

Given a mixed strategy $s^1$, we can find $b^1$ as follows:

- $s^1(L_1, L_2) + s^1(L_1, R_2) = p$;
- $\frac{s^1(R_1, L_2)}{s^1(L_1, L_2) + s^1(L_1, R_2) + s^1(R_1, R_2)} = q$.

Note that as a corollary of Kuhn’s Theorem, every EFG with perfect recall has a NE in behavioral strategies. Indeed, the theorem says that it doesn’t matter whether we think of behavior in terms of mixed or behavioral strategies if every player has perfect recall.

### 2.6 Subgame Perfect Nash Equilibrium

This equilibrium concept is motivated by the observation that NE sometimes does not make intuitive predictions in extensive form games.
**Example:** As motivation, consider the following game. There are two players: Firm E (Entrant) and Firm I (Incumbent). Firm E chooses whether to enter a market or not. If it does not enter, the payoffs are 0 to Firm E and 2 to Firm I (Firm I gets all the profit). If Firm E enters, Firm I has two choices: to accommodate or fight. If Firm I accommodates, it gets a payoff of 1 while firm E gets a payoff of 2. If Firm I fights, Firm E gets a payoff of -3 and Firm I gets a payoff of -1.

![Figure 1: Example: Market entry game](image)

The normal form of this game is given below:

<table>
<thead>
<tr>
<th></th>
<th>Accomodate</th>
<th>Fight</th>
</tr>
</thead>
<tbody>
<tr>
<td>In</td>
<td>2,1</td>
<td>-3,-1</td>
</tr>
<tr>
<td>Out</td>
<td>0,2</td>
<td>0,2</td>
</tr>
</tbody>
</table>

The two pure strategy equilibria of this game are (In, Accomodate) and (Out,Fight). While the former equilibrium is reasonable, the latter one is not. This is because Player E knows that if he plays In, there is no way that Player I will decide to fight.

**Example (Finitely Repeated Prisoner’s Dilemma):** Consider the following Prisoner’s Dilemma game played twice:

<table>
<thead>
<tr>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>-2,-2</td>
</tr>
<tr>
<td>D</td>
<td>-1,-10</td>
</tr>
</tbody>
</table>

For an exercise, draw an extensive form version of this game. Show that the player who makes the last move is better off choosing D regardless of what was chosen in the preceding levels of the game. Argue that this leads to both players defecting whenever they are given a choice to move.
2.7 Backward Induction

Definition: An EFG is of perfect information if for every player $i$ and every $u \in U^i$, $u$ is a singleton.

Notice that any game of perfect information can be solved using the following procedure, called backward induction:

1. Look at immediate predecessors of the final nodes.
2. Each such node has a player controlling it. Choose the action that gives him the largest payoff (break ties arbitrarily).
3. Replace the node with a final node having utility for each player equal to the utility induced by the action chosen in Step 2.
4. Repeated the procedure in steps 1-3 in the new game until only one node is left.

The chosen actions describe a pure strategy profile. This strategy profile is an equilibrium.

Theorem (Zarmelo): Every game of perfect information has an equilibrium in pure strategies.

2.8 Subgame Perfect Equilibria

Definition (Subgame): A subgame is the game beginning at every node if it really is a game, i.e. if every player knows the game.

Formally, let $K_x = \{y : x \succeq y\}$. This is the set of nodes following $x$.

- $\succeq_x$ is the restriction of $\succeq$ to $K_x$.
- $P^i_x = P^i \cap K_x$.
- $Z_x = Z \cap K_x$.
- $u^i_x(\cdot) = u^i(\cdot)$ defined on $Z_x$. 

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A subtree is a subgame if whenever $x_1, x_2 \in u$ for some $u \in U^i$ and $x_1 \in K_x$, it is also the case that $x_2 \in K_x$.

**Definition (SPNE):** A behavioral strategy profile $b$ is a Subgame Perfect Nash Equilibrium (SPNE) is its restriction to the subgame $G_x$ is an equilibrium in $G_x$ for all subgames $G_x$ of $G$.

Note: SPNE is clearly a Nash Equilibrium, so this is another example of a NE refinement.

**Theorem:** SPNE exists.

**Sketch of proof:** To prove the theorem, we can construct a SPNE of any extensive form game $G$ using the following backward induction-like procedure:

1. Take all the minimal subgames of $G$ (minimal = has no subgames).
2. For each of these subgames, find its behavioral NE strategy.
3. Replace the original game with the game where the minimal subgames are removed and the nodes are assigned expected payoffs from the associated behavioral NE.
4. Repeat.

### 2.9 Sequential Equilibrium

SPNE has the problem that some strategies it prescribes cannot be supported by any beliefs (see example in class).

To deal with this problem, we define a new equilibrium concept which includes beliefs explicitly in the definition.

**Definition:** A belief system $\mu$ is a map $\mu : X \rightarrow [0, 1]$ s.t. $\sum_{x \in u} \mu(x) = 1$ for all $i, u \in U^i$. **Definition:** An assessment is a tuple $(b, \mu)$ where $b \in B$ and $\mu$ is a belief system.

**Definition:** The set of $\epsilon$-fully mixed behavioral strategies of player $i$ is

$$B^i_\epsilon = \{ b^i \in B^i : b^i_u(c) \geq \epsilon \ \forall c \in C_u, \forall u \in U^i \}. $$
Definition (Consistent Assessments): An assessment \((b, \mu)\) is consistent if there exists a sequence \((b_n, \mu_n)\) s.t.

1. \(b_n\) is fully mixed \(\forall n\);
2. \((b_n, \mu_n) \rightarrow (b, \mu)\);
3. \(\mu_n(x|u) = \frac{Pr_b(x)}{Pr_b(u)} \forall n, x, u\).

In words, the third condition says that each \(\mu_n\) is derived from \(b_n\) using Bayes’ Rule.

Example: Consider the game below.

Consider \(b = ((1, 0, 0), (0, 1))\) and \(\mu\) s.t. the distribution over Player 2’s nodes is \((\alpha, 1 - \alpha)\). The claim is that this assessment is consistent for any \(\alpha \in (0, 1)\). Let \(b_n^1 = (1 - \epsilon_n, \alpha \epsilon_n, (1 - \alpha) \epsilon_n)\) and \(b_n^2 = (\epsilon_n, 1 - \epsilon_n)\). Note that \(\mu_n(x|u) = \frac{\alpha \epsilon_n}{\epsilon_n} = \alpha\) is the belief of Player 2 about his first node derived using Bayes’ Rule.

Definition (Sequential Rationality): An assessment \((b, \mu)\) is sequentially rational if

\[ E(u^i(b^i, b^{-i}; \mu)|u) \geq E(u^i(d^i, b^{-i}; \mu)|u) \quad \forall i, u \in U^i, d^i \in B^i. \]

Definition (Sequential Equilibrium): An assessment \((b, \mu)\) is a sequential equilibrium if it is consistent and sequentially rational.

Example: Consider the game shown above. The assessment \((b, \mu)\), where \(b = ((1, 0, 0), (0, 1))\) and \(\mu\) s.t. the distribution over Player 2’s nodes is \((\alpha, 1 - \alpha)\), is a sequential equilibrium if \(\alpha \leq 1/2\). Question: Can you find other sequential equilibria?
3 Repeated Games

The big message of the theory of repeated games has two parts. The message is:

- When the game is repeated, you get a lot of “new” Nash Equilibria (a loss of predictive of power),
- When the game is repeated, it’s possible to sustain more cooperation than when the game is played once.

Both parts of the message are captured by a set of results that are referred to as the “folk theorems.” Let’s spend some time trying to understand what these theorems say.

In a repeated game,\(^1\) the fundamental building block is the stage game, which is a normal form game \( \{ I, \{ A^i \}, \{ g^i \} \} \). This stage game can be played finitely or infinitely many times.

**Finitely repeated games**

This model is appropriate if the termination time is well-known by all the players.

**Infinitely repeated games**

This model is appropriate if the termination time is random and not known by the players precisely.

Let \( a_t = (a^1_t, ..., a^I_t) \) be the actions played in period \( t \). The game begins in period 0 with null history \( h_0 \). For \( t \geq 1 \), \( h_t = (a_0, a_1, ..., a_t) \). Let \( H_t = A^t \) be the space of period-\( t \) histories.

**Definition (pure strategies):** A pure strategy of player \( i \) specifies a map

\[
\sigma^i_t : H^t \to A^i
\]

for every period. Thus, a pure strategy is \( (\sigma^i_0, \sigma^i_1, \sigma^i_2, ...) \).

**Definition (behavioral strategies):** A behavioral strategy of player \( i \) specifies a map

\[
b^i_t : H^t \to S^i
\]

\(^1\) These notes are largely based on Chapter 5 of Fudenberg and Tirole.
for every period. Thus, a behavioral strategy is \((b^0_i, b^1_i, b^2_i, ...).\)

Note that a subgame begins at every period of play. We will use this fact later to think about subgame perfect equilibria.

### 3.1 Infinitely Repeated Games

We focus on the case where players discount future utilities with a discount factor \(\delta < 1.\)

Given a profile of behavioral strategies \(b\), player \(i\)’s utility is

\[
(1 - \delta)E_b[g^i(b_0(h_0)) + \delta g^i(b_1(h_1)) + \delta^2 g^i(b_2(h_2))...] = (1 - \delta)E_b \left[ \sum_{t=0}^{\infty} \delta^t g_i(b_t(h_t)) \right].
\]

I.e., it is the expected discounted stream of payoffs with \(b\) multiplied by \(1 - \delta\). The reason for this multiplication is that we want the units to be in terms of the stage game payoffs, i.e. we are in some sense thinking of the average payoffs.

Example Consider an infinitely repeated game where both player have a discount factor \(\delta < 1\) and the stage game is the following battle of the sexes game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>3,1</td>
<td>0,0</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>1,3</td>
</tr>
</tbody>
</table>

Assume Player 1 chooses B in every period and Player 2 chooses R in every period. Then, Player 1’s discounted payoff is:

\[
1 + \delta^1 + \delta^2 + \delta^3 + \delta^4 + ... = \frac{1}{1 - \delta}.
\]

Multiplied by \(1 - \delta\), we get 1, Player 1’s average payoff in the game.

Observation: Let \(\hat{\alpha}\) be a Nash Equilibrium of the stage game (a “static equilibrium”). Then, the strategy profile in which every player \(i\) plays \(\hat{\alpha}^i\) in every period of the repeated game is a Nash Equilibrium of the repeated game. Moreover, if the game has \(m\) static equilibria \(\{\hat{\alpha}_j\}_{j=1}^m\), then any map from time periods to indices \(\{1, ..., m\}\) implies a Nash Equilibrium of the repeated game in which every player plays \(\hat{\alpha}^i_{j(t)}\) in period \(t\).
The reason for the observation is: The repeated game strategies described above do not depend on what any player has done in any previous period. Take, for instance, the example where every player $i$ plays $\alpha^i$ in every period of the repeated game. When player $i$ finds himself in period $t$, he asks himself the question: “What is the best thing for me to do now, given that the other players are playing according to $\alpha^{-i}$?” Since $\alpha$ is a Nash Equilibrium, the answer is to choose $\alpha^i$.

What this observation shows is that turning a static game into a repeated game does not make the set of equilibria smaller. In fact, it makes it bigger, as we will now see.

### 3.2 Folk Theorems for Repeated Games

The folk theorems say that any feasible, individually rational payoffs can be sustained in equilibrium in a repeated game if players are sufficiently patient. To understand what this means, we need to understand the words “feasible” and “individually rational”.

**Individually rational payoffs**

**Definition (Minimax value):** Define player $i$’s reservation utility or minimax value as

$$v^i = \min_{s^{-i}} \max_{s^i} g^i(s^i, s^{-i}).$$

In words, this is the lowest payoff that the other players can hold player $i$ to, assuming that he correctly foresees what they do and plays a best response. Let $(m^i, m^{-i})$ be the strategy profile so that $g^i(m^i, m^{-i}) = v^i$.

**Example:** Find the minimax values of Player 1 and Player 2 in the following game:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>-2,2</td>
<td>1,-2</td>
</tr>
<tr>
<td>M</td>
<td>1,-2</td>
<td>-2,2</td>
</tr>
<tr>
<td>B</td>
<td>0,1</td>
<td>0,1</td>
</tr>
</tbody>
</table>

**Observation:** Player $i$’s payoff is at least $v^i$ in any static equilibrium and in any Nash Equilibrium of the repeated game.
**Proof:** In a static equilibrium \( \hat{\alpha}, \hat{\alpha}^i \) is a best response to \( \alpha^{-i} \). Thus,
\[
g^i(\hat{\alpha}^i, \alpha^{-i}) \geq g^i(m^i_i, \alpha^{-i}) \geq \min_{s^{-i}} g^i(m^i_i, s^{-i}) = g^i(m^i_i, m^{-i}).
\]
This shows the first part of the observation.

For the second part of the observation, consider a Nash Equilibrium \( \hat{\alpha} \) of the repeated game. One feasible (not necessarily optimal) strategy for player \( i \) is to maximize \( g^i(s^i, \hat{\alpha}^{-i}) \) in every period. Now,
\[
\max_{s^i} g^i(s^i, \hat{\alpha}^{-i}) \geq \min_{s^{-i}} \left[ \max_{s^i} g^i(s^i, s^{-i}) \right] = \bar{v}^i.
\]
This shows the second part of the observation.

**Feasible payoffs**

We define the set of feasible payoffs as the convex hull of the payoffs in the stage game:
\[
V = \text{convex hull}\{v | \exists a \in A \text{ with } g(a) = v\}.
\]
Remember that not all of these payoffs are feasible in the stage game! Some convex combinations of stage game payoffs correspond to correlated equilibrium payoffs, that you cannot get with Nash. In repeated games, though, the story is different. You can get things in the convex hull even without correlated equilibria (assuming that the players are sufficiently patient).

**Definition (Feasible, Individually Rational Payoffs):** The set of feasible, strictly individually rational payoffs is
\[
\{v \in V | v^i > \bar{v}^i \ \forall i\}
\]

**Example:** In the game above, the set of feasible, individually rational payoffs is given in the following picture:
**Theorem (Folk Theorem):** For every feasible payoff vector with $v^i > v^j$ for all players $i$, there exists $\delta < 1$ such that for all $\delta \in (\delta, 1)$, there is a Nash Equilibrium $G(\delta)$ of the repeated game with payoffs $v$.

**Proof** Assume first that there is a pure strategy profile $a$ s.t. $g(a) = v$. Consider the following strategy for each player $i$:

- Start by playing $a_i$;
- Continue playing $a_i$ as long as the previous action was $a$ or the previous action differed from $a$ in two or more components. If in some previous period period player $i$ was the only one not to follow the profile $a$, then each player $j$ plays $m^i_j$ for the rest of the game.

Can anyone gain by deviating from this strategy profile? Consider the player who deviates in period $t$, assuming everyone else is being a good boy. In the period the player deviates, he gets AT MOST

$$(1 - \delta^t) v^i + \delta^t \left[ (1 - \delta) \max_a g^i(a) + \delta v^i \right] = (1 - \delta^t) v^i + \delta^t (1 - \delta) \max_a g^i(a) + \delta^{t+1} v^i.$$ 

For a $\delta$ close to 1, the deviating player’s payoff is clearly less than $v^i$ (since $v^i < v^j$). This finishes the proof for the case where $v$ can be attained in pure strategies.
Now consider the case where $v$ cannot be attained in pure strategies. Here, our proof will make use of public randomizations. A public randomization is a publicly observed coin flip $w$ that the players can use to correlate their actions. While the Folk Theorem can also be proved without public randomizations, we use them because it makes the proof easier.

Formally, let $\{\omega_1, \omega_2, \ldots\}$ be a sequence of uniform draws from the interval $[0, 1]$. The history is now $h_t = (a_0, a_1, \ldots, a_{t-1}, \omega_0, \omega_1, \ldots, \omega_{t-1})$. As before, a pure strategy is a sequence of maps from histories to actions.

Let $v$ be a feasible, strictly individually rational payoff vector that cannot be attained in pure strategies. Let $a(\omega)$ be a public randomization with expected payoffs $v$. Consider the following strategy:

- Start by playing according to $a(\omega)$;
- Continue playing according to $a(\omega)$ as long as everyone played according to $a(\omega)$ in the past, or the number of deviating players was two or greater. If in some previous period period player $i$ was the only one not to play according to $a(\omega)$, then each player $j$ plays $m_j^i$ for the rest of the game.

Define $\bar{\delta}$ as follows:

$$(1 - \bar{\delta}) \max_a g(a) + \bar{\delta}v_i = (1 - \bar{\delta}) \min_a g(a) + \bar{\delta}v_i.$$  

For $\delta > \bar{\delta}$, if some player $i$ deviates in some period $t$ he gets at most:

$$(1 - \delta^t)v^i + \delta^t \left[ (1 - \delta) \max_a g^i(a) + \delta v^i \right] \leq (1 - \delta^t)v^i + \delta^t \left[ (1 - \delta) \min_a g^i(a) + \delta v^i \right] \leq v^i$$

This finishes the proof.

### 3.2.1 Example (Repeated Cournot Oligopoly)

Consider a two firm Cournot Oligopoly in which the demand curve is given by $p(q)$ with $p(q) \to 0$ as $q \to \infty$ and each firm faces a per-unit cost $c$. The Folk Theorem says that for a large enough discount factor the firms can collude to each produce of
one half of the monopoly output by threatening each other to produce competitive equilibrium quantities (i.e., minmaxing. Work out the details for the homework).

There is a potential problem here. Imagine now that the costs of the two firms are different: Firm $i$ faces a per-unit cost of $c_i$. In principle, it’s possible that the quantity $q_j$ that minimaxes firm $i$ is such that price exceeds $c_j$. This raises the question of whether firm $j$ would want to commit to following the punishment path. More generally, the issue is that the equilibrium strategies described above are not subgame perfect.

Is it possible to sustain feasible, individually rational payoffs using subgame perfect strategies?

**Theorem (Nash Threat Folk Theorem):** Let $\alpha^*$ be a static Nash Equilibrium of the stage game with payoffs $e$. For every feasible payoff vector with $v^i > e^i$ for all players $i$, there exists $\delta < 1$ such that for all $\delta \in (\delta, 1)$, there is a Subgame Perfect Nash Equilibrium $G(\delta)$ of the repeated game with payoffs $v$.

**Proof:** The proof of this version of the Folk Theorem looks very similar to the proof we studied above. Let’s assume first that there is an action profile $a$ s.t. $g(a) = v$. Consider the strategy in which each player plays $a^i$ as long as the other players do it. If one of the players deviates, all others play $\alpha^*$. The deviating player gets at most

$$(1 - \delta^i)v^i + \delta^i \left[ (1 - \delta) \max_a g^i(a) + \delta e^i \right].$$

Since $e^i < v^i$, for large enough $\delta$, the deviation payoff is less than $v^i$. If there is no action profile with $g(a) = v$, we use public randomizations as before (end of proof).

This version of the Folk Theorem is somewhat more reassuring: It tells us that Cournot oligopolists can collude by threatening each other with static Cournot quantities. These are at least static Nash Equilibria, so commitment is not an issue.
3.3 Application: Time-Consistent Monetary Policy

We can use repeated games to study inflation. In particular, consider a game between a central bank and firms. The central bank sets an inflation rate $\pi$ to maximize the following function:

$$U(\pi, y) = -c\pi^2 - (y - y^*)^2.$$ 

The parameter $c > 0$ reflects how important it is for the central bank to worry about inflation. $y^*$ is the efficient level of output, and the central bank loses utility if output is away from efficiency.

Firms simply want to have correct expectations; their payoff is given by

$$-(\pi - \pi^e)^2$$

Actual output is a function of both the efficient level of output and inflation. Intuitively, is there is surprise inflation, real wages are lower, and employers will expand employment and output. We assume that in absense of surprise inflation output is below the efficient level ($b < 0$) and express output as follows:

$$y = by^* + d(\pi - \pi^e).$$

The parameter $d > 0$ captures the effect of surprise inflation on real wages as described above.

Plugging into the central bank’s utility function, we get

$$U(\pi, \pi^e) = -c\pi^2 - [(b - 1)y^* + d(\pi - \pi^e)]^2$$

The bank’s best response is to set

$$\hat{\pi}(\pi^e) = \frac{d[(1 - b)y^* + d\pi^e]}{c + d^2}$$

In a (static) equilibrium, firms hold rational expectations:

$$\pi^e = \hat{\pi}(\pi^e)$$

Plugging in and solving for $\pi^e$, we get

$$\hat{\pi}(\pi^e) = \pi^e = \frac{d(1 - b)}{c}y^* = y^*.$$
where \( \pi^s \) stands for the stage game inflation rate.

Thus, we get positive inflation in (static) equilibrium. The central bank’s equilibrium payoff is

\[-c(\pi^s)^2 - (y^*(b - 1))^2\]

The central bank, however, would be better off having no inflation. Indeed, if firms hold rational expectations, \( \pi = 0 \) maximizes the central bank’s payoff with payoff

\[-(y^*(b - 1))^2\]

Now assume that the central bank and the firms are playing a repeated game with discount factor \( \delta \). Consider the following strategy profile:

- Central bank starts by setting \( \pi = 0 \) and the firms start by setting \( \pi^s = 0 \)
- Continue choosing \( \pi = 0 \) and \( \pi^s = 0 \) as long as both parties have done so in the previous period
- If \( \pi^e \neq 0 \) in the previous period, the central bank chooses \( \pi = \pi^s \) and keeps choosing it forever
- If \( \pi \neq 0 \) in the previous period, the firms choose \( \pi = \pi^s \) and keep choosing it forever

Is this strategy profile an equilibrium? First, note that as long as the central bank is choosing \( \pi = 0 \), the firms best respond by setting \( \pi^e = 0 \). I.e., deviations by the firms are unprofitable. Now let’s consider a deviation by the central bank. The deviation is unprofitable if

\[
\frac{1}{1 - \delta} U(0, 0) \geq U(\hat{\pi}(0), 0) + \frac{\delta}{1 - \delta} U(\pi^s, \pi^s).
\]

This reduces to

\[\delta \geq \frac{c}{2c + d^2}\]

Notice that this is decreasing in \( d \) and increasing in \( c \). Why?

Intuitively, \( d \) has two effects. On the one hand, it makes the benefit of surprise inflation greater, which makes it tempting for the central bank to raise \( \pi \). On the
other hand, it increases $\pi^*$, which makes the potential punishment greater. That $\delta$ is decreasing in $d$ means that the latter effect outweighs the former.

Similarly, $c$ has two effects. On the one hand, it makes the benefit of surprise inflation smaller. On the other hand, it decreases $\pi^*$, which makes the equilibrium punishment seem smaller. That $\delta$ is increasing in $c$ means that the latter effect outweighs the former.
3.4 Imperfect Monitoring

We will now consider a class of repeated games with imperfect monitoring. These are games in which players don’t observe each other’s actions. Instead, they observe some noisy (imperfect) information about what the other players are doing. First, consider a repeated game where player i’s payoff \( r^i(a^i, y) \) in every period depends on his own action \( a^i \) and a public signal \( y \). The other player’s actions enter through the distribution \( \pi_a(y) \) over the public signal. In every period, the player’s expected payoff from the action profile \( a \) is given by:

\[
g^i(a) = \sum_y \pi_a(y) r^i(a^i, y)
\]

We will restrict our attention to strategies that are functions of the observed signals. A (public) history is the history of \( y \)’s observed in previous periods. Thus,

\[ h_t = (y_0, y_1, y_2, ... ) \]

A (behavioral) strategy of player \( i \) specifies a map

\[ b^i_t : H^t \rightarrow S^i \]

for every period. Thus, a behavioral strategy is \( (b^i_0, b^i_1, b^i_2, ...) \), a sequence of maps from histories to mixed actions.

Example (Noisy Prisoner’s Dilemma)

Consider an infinitely repeated prisoner’s dilemma (like the one we studied before) with the following slight modification. In every period, there is a probability \( \epsilon \) of each player independently making a mistake. The signal now is the observed action profile, but it depends noisily on the chosen action profile as follows:

\[
\begin{align*}
\pi_{C,C}(C, C) &= (1 - \epsilon)^2 \\
\pi_{C,C}(C, D) &= \pi_{C,C}(D, C) = \epsilon(1 - \epsilon) \\
\pi_{C,C}(D, D) &= \epsilon^2
\end{align*}
\]

Thus, \( H = A \) and \( h_t \) is the sequence of observed actions, which may be different from the intended actions. This is a repeated game with imperfect monitoring.
From now on, we will change the environment slightly. We will assume that the underlying game is some normal form game \( G \). The players’ payoffs are given by the payoffs in the normal form game, but these payoffs are unobserved. All that the players observe is the history of the public signals. We will focus on **perfect public equilibria**. In words, these are equilibria in strategies *that only depend on histories of the public signal*. Our definition will also imply subgame perfection:

**Definition** A profile \( b = (b^1, ..., b^I) \) is a **perfect public equilibrium** if

1. Each \( b^i \) is a public strategy;
2. For each \( t, h_t \), the strategies yield a Nash equilibrium from that date on.

Let’s study what we can achieve in perfect public equilibria in the following simple example:

**Example 2**

Depending on whether she chose to cooperate (\( C_i \)) or defect (\( D_i \)), subject \( i \)'s stock of points increases by \( g_{it} \) in period \( t \) according to the following table:

<table>
<thead>
<tr>
<th></th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td>15, 15</td>
<td>0, 20</td>
</tr>
<tr>
<td>( D )</td>
<td>20, 0</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

Instead of observing her partner’s actions, each player sees a public signal that could go up with probability \( p(a_t) \) or down with probability \( 1 − p(a_t) \), with \( p(a_t) \) determined as in the table below:

<table>
<thead>
<tr>
<th></th>
<th>( C )</th>
<th>( D )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td>( \frac{3}{4} ), ( \frac{1}{4} )</td>
<td>( \frac{1}{2} )</td>
</tr>
<tr>
<td>( D )</td>
<td>( \frac{1}{2} ), ( \frac{1}{2} )</td>
<td></td>
</tr>
</tbody>
</table>

**Proposition (Bound on Public Equilibrium Payoffs)** Let \( \gamma(\delta) = \max\{v_1 + v_2 : v \in E(\delta)\} \), where \( E(\delta) \) is the set of public equilibrium payoff vectors of the game with discount factor \( \delta < 1 \). Then, \( \gamma(\delta) \leq 20 \) for every \( \delta \).

An illustration of this proposition is given below:
Proof: The proof follows a basic argument by Fudenberg, Levine and Maskin (1994). For a contradiction, assume that \( \gamma > 20 \). Choose \( v \in E(\delta) \) such that \( v_1 + v_2 = \gamma \). Player \( i \)'s utility is given by

\[
v_i = (1 - \delta)u_i + \delta(pw_i^+ + qw_i^-),
\]

where \( w_i^+ \) and \( w_i^- \) denote the continuation payoffs of player \( i \) after a good and a bad signal, respectively, \( u_i \) is player \( i \)'s expected utility today and \( p \) and \( q \) are, respectively, the probabilities of a good and a bad signal today. Since \( \gamma > 20 \), it must be the case that the probability of both players cooperating is greater than zero after some history. Let \( \mu_j \), where \( j \neq i \), denote player \( j \)'s probability of defection. It will be incentive compatible for player \( i \) to cooperate if

\[
(1 - \delta)(1 - \mu_j)15 + \delta[(1 - \mu_j)p_2 + \mu_j p_1)]w_i^+ + ((1 - \mu_j)q_2 + \mu_j q_1)w_i^- \geq (1 - \delta)(1 - \mu_j)20 + 2\mu_j + \delta[(1 - \mu_j)p_1 + \mu_j p_0)]w_i^+ + ((1 - \mu_j)q_1 + \mu_j q_0)w_i^-.
\]

Write \( \Delta p = (1 - \mu_j)(p_2 - p_1) + \mu_j(p_1 - p_0) \). Rearranging yields:

\[
(1 - \delta)[5(1 - \mu_j) + 2\mu_j] \leq \delta \Delta p(w_i^+ - w_i^-),
\]
that is, current utility gains from deviating are outweighed future losses in continuation payoffs. Since $p_2 - p_1 = 1/4$ and $p_1 = p_0$, this inequality yields the following upper bound on $w_i^-$:

$$w_i^- \leq w_i^+ - \frac{1 - \delta}{\delta} \frac{5(1 - \mu_j) + 2\mu_j}{\Delta p} = w_i^+ - \frac{1 - \delta}{\delta} \frac{5(1 - \mu_j) + 2\mu_j}{(1 - \mu_j)/4} \leq w_i^+ - \frac{20}{\delta} \frac{1 - \delta}{\delta}$$

Substituting into the previous expression for $v_i$,

$$v_i \leq (1 - \delta)u_i + \delta \left[ w_i^+ - 20q \frac{1 - \delta}{\delta} \right].$$

Therefore,

$$v_1 + v_2 \leq (1 - \delta) \left[ 30 - 40q \right] + \delta \gamma.$$ 

Since $q \geq 0.25$ and $v_1 + v_2 = \gamma$ by hypothesis, it follows that $\gamma \leq 20$, as claimed.

QED

3.5 Abreu, Milgrom, and Pearce (1990)

Abreu, Milgrom, and Pearce (1990) showed that the bound on public equilibrium payoffs (described above) can be overcome in two ways: by delaying and information and by varying the time period over which players’ actions are held fixed. Let’s examine the two elements of their proposition in detail.

How information delay can help

Suppose that, instead of the signal arriving every period, it was possible to lump the information in such a way that the signal only arrived at the end of every $T$-period block. Abreu, Milgrom, and Pearce (1990) show how players can improve upon a welfare of 20 by delaying information this way. Consider the following strongly symmetric strategies, to be called AMP block strategies. Every player cooperates for $T$ periods. At the end of the $T$-period block, the $T$ public signals for each period in the block arrive to the players. If every signal was bad then continuation play consists of mutual defection henceforth with some probability $\alpha$. Otherwise, they continue to cooperate for the next block with the same contingency.

The probability of $T$ consecutive bad signals equals $q_2^T$ in equilibrium, that is, assuming mutual cooperation throughout the block. A player’s lifetime utility under
Figure 2: AMP block strategies ($b^T = T$ bad signals)

this strategy profile is therefore given by

$$v = (1 - \delta^T) 15 + \delta^T [(1 - q_2^T) v + q_2^T ((1 - \alpha) v + 2 \alpha)].$$

Rearranging,

$$v = 15 - \frac{\delta^T}{1 - \delta^T} q_2^T \alpha (v - 2).$$

(3.1)

Discouraging a deviation in the very first period of the block requires that the utility gained from defecting, $(1 - \delta)5$, be outweighed by the associated loss in continuation payoff. This is given by the change in probability of punishment from the one-period deviation, $q_2^T - 1(q_2 - q_2^T)$, times the opportunity cost of punishment, $\delta^T \alpha (v - 2)$. Since $q_1 - q_2 = .25 = q_2$, this incentive constraint may be written as

$$(1 - \delta)5 \leq \delta^T q_2^T \alpha (v - 2).$$

(3.2)

A key insight behind the welfare properties of AMP block strategies is that discouraging one deviation discourages all others, as the next result shows. The intuition for it is this. The gains from deviating grow linearly, whereas the costs grow exponentially in the number of deviations. Therefore discouraging one deviation discourages them all.

**Proposition (Equilibrium in AMP Strategies):** If the AMP block strategies above discourage a deviation in any single period of a block then they discourage every deviation, that is, they constitute an equilibrium.

**Proof:** Assume that (3.2) holds. If a player chooses to deviate for $\tau$ periods, the utility gained from such a deviation is clearly bounded above by $(1 - \delta)^{5\tau}$, since
this bound ignores discounting of future deviation gains. In other words, deviation gains are linear in the number of deviations. On the other hand, punishment costs grow exponentially in the number of deviations. Indeed, the opportunity cost of punishment remains $\delta^T \alpha(v - 2)$, but the change in punishment probability from $\tau$ deviations becomes

$$q_2^{T-\tau}(q_1^{\tau} - q_2^{\tau}) = q_2^T \left[ \left( \frac{q_1}{q_2} \right)^\tau - 1 \right],$$

which, since $q_1 > q_2$, clearly grows exponentially with $\tau$. Now, by the Binomial Theorem, $(q_1/q_2)^\tau - 1 \geq \tau[(q_1/q_2) - 1] = \tau$, so the change in punishment probability is bounded below by $q_2^T \tau$. Therefore, the following inequality implies that $\tau$ deviations are discouraged:

$$(1 - \delta)\delta^\tau \leq \delta^T q_2^T \tau \alpha(v - 2).$$

But this is just (3.2). The claim now follows because $\tau$ was arbitrary. QED

Consider maximizing $v$, the equilibrium payoffs above, with respect to $\alpha$ such that the AMP block strategies above remain an equilibrium. At an optimum, the incentive constraint (3.2) must bind, since otherwise by (3.1) we would be able to feasibly lower $\alpha$ further and increase $v$, contradicting optimality. If (3.2) binds then the maximum value of $v$ equals

$$v = 15 - 5 \frac{1 - \delta}{1 - \delta^T}.$$

On the other hand, feasibility requires that $\alpha \leq 1$, since it is a probability. Substituting for $v$ and this inequality in (3.2) and rearranging gives

$$5(1 - \delta) \left[ \frac{1}{1 - \delta^T} + \frac{1}{(\delta q_2)^T} \right] \leq 13. \quad (3.3)$$

This inequality places a restriction on the exogenous parameters of the game for the strategy profile above to be an equilibrium. Abreu, Milgrom, and Pearce (1990) used a version of this bound to argue a result along the following lines.

**Proposition (Overcoming the bound with delay)** For every block length $T \in \mathbb{N}$, there exists $\delta < 1$ sufficiently large that the strategies above constitute an equilibrium. Moreover,

$$\lim_{T \to \infty} \lim_{\delta \to 1} v = 15.$$

**Proof:** Fix $T \in \mathbb{N}$. As $\delta \to 1$, the left-hand side of (3.3) tends to $1/T$ by l’Hopital’s rule, which is less than or equal to 1. Hence, there exists $\delta < 1$ sufficiently
large that (3.3) holds, so the candidate equilibrium strategies above are indeed an equilibrium. Finally, by l’Hopital’s rule, it follows that

\[ v \to 15 - \frac{5}{T} \] as \( \delta \to 1 \).

Finally, it is now clear that \( v \to 15 \) as \( T \to \infty \), as claimed.

Note that \( 15 + 15 > 20 \). This shows that the public equilibrium bound on payoffs can be overcome with delay. QED

**How bounded rationality can help**

Bounded rationality can also help players overcome the bound on public equilibrium payoffs. Let \( \tau = 2 \) denote the number of periods that a player is unable to change her action, and consider the following strongly symmetric strategies. Every player cooperates for \( \tau \) periods. At the end of the \( \tau \)-period block, the players consider cooperating if anything other than \( \tau \) bad signals is observed. If \( \tau \) bad signals are observed, the players switch to defection with some probability \( \alpha \).

The probability of 2 consecutive bad signals equals \( q_2^2 \) in equilibrium. A player’s lifetime utility under this strategy profile is therefore given by

\[ v = (1 - \delta^2)15 + \delta^2[(1 - q_2^2)v + q_2^2((1 - \alpha)v + 2\alpha)]. \]

Rearranging,

\[ v = 15 - \frac{\delta^2}{1 - \delta^2}q_2^2\alpha(v - 2). \] (3.4)

Discouraging a deviation requires that the utility gained from defecting, \((1 - \delta^2)5\), be outweighed by the associated loss in continuation payoff. Thus, the incentive constraint is

\[ (1 - \delta^2)5 \leq \delta^2\alpha(q_1^2 - q_2^2)(v - 2). \] (3.5)

Consider maximizing \( v \) with respect to \( \alpha \) such that the trigger strategies above remain an equilibrium. At an optimum, the incentive constraint (3.5) must bind, since otherwise by (3.4) we would be able to feasibly lower \( \alpha \) further and increase \( v \), contradicting optimality. If (3.5) binds then the maximum value of \( v \) equals

\[ v = 15 - \frac{5}{\left(\frac{q_1^2}{q_2^2}\right)^2 - 1} = 15 - 5/3 \approx 13.33 > 10. \]
It is easy to check that the feasibility constraint, that is, $0 \leq \alpha \leq 1$, is satisfied in our example.
4 Mechanism Design

A mechanism design problem arises when we want some outcome \( x \in X \) implemented as a result of a game played by some agents whose preferences we don’t know.

We will consider environments where agents’ preferences are indexed by a vector \( \theta \in \Theta \). We will call \( \theta_i \) agent \( i \)’s type. Each realization of \( \theta_i \) induces a utility function over \( X \):

\[
u_i(\cdot, \theta_i) : X \rightarrow \mathbb{R}
\]

**Definition** Formally, a mechanism \( \Gamma = (M, g) \) is a collected of strategy sets \((M^1, ..., M^N)\) and an outcome function \( g : M \rightarrow X \).

Note that a mechanism together with \( \theta \) (a profile of types) induces a normal form game in which each agent \( i \) chooses from \( M_i \) with a utility function \( u_i(\cdot, \theta_i) \).

Motivating Example: King Solomon’s Dilemma

Consider a problem with

- Two players, Anna and Beth;
- Two states of the world, \{Anna is the true mother, Beth is the truth mother\} = (\( \alpha \), \( \beta \));
- Four outcomes, \{Baby to Anna, Baby to Beth, Cut baby, Death to everyone\}.

If the state of the world is \( \alpha \), the preferences are:

\[
A \succ_{Anna} B \succ_{Anna} C \succ_{Anna} D
\]

\[
B \succ_{Beth} C \succ_{Beth} A \succ_{Beth} D
\]

If the state of the world is \( \beta \), the preferences are:

\[
A \succ_{Anna} C \succ_{Anna} B \succ_{Anna} D
\]

\[
B \succ_{Beth} A \succ_{Beth} C \succ_{Beth} D
\]

King Solomon (the mechanism designer) wants to implement the social choice function that gives the baby to Anna when the state of the world is \( \alpha \) and gives the
baby to Beth when the state of the world is $\beta$. Formally, this social choice function is defined as follows: $f^*(\alpha) = A$, $f^*(\beta) = B$.

Consider the class of mechanisms where the strategy space faced by both women is \{Scream, Stand\}. It remains to specify the outcome function. Consider the following mechanism:

<table>
<thead>
<tr>
<th>Beth screams</th>
<th>Anna screams</th>
<th>Beth stands</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>B</td>
<td>C</td>
</tr>
</tbody>
</table>

This induces the following normal form game if the state is $\alpha$ (we assign 4 to the highest valued outcome, 3 to the second highest, etc.):

<table>
<thead>
<tr>
<th>Beth screams</th>
<th>Anna screams</th>
<th>Beth stands</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1,1)</td>
<td>(4,2)</td>
<td></td>
</tr>
<tr>
<td>(3,4)</td>
<td>(2,3)</td>
<td></td>
</tr>
</tbody>
</table>

This game has two Nash equilibria: (Anna screams, Beth stands) and (Anna stands, Beth screams). The set of Nash equilibrium outcomes is $\{A, B\}$. Thus, when the state of the world is $\alpha$, one of the equilibrium outcomes gives the baby to Beth. Therefore, this mechanism does not (strongly) implement King Solomon’s social choice function in Nash equilibrium.

**Definition (Strong Nash Implementation):** A mechanism $\Gamma = (M, g)$ strongly implements the social choice function $f$ in Nash equilibrium if $\{f(\theta)\} = g(m^*(\theta))$, where $m^*(\theta)$ is any Nash Equilibrium of the game $(\Gamma, \theta)$.

The question is: Is there any mechanism King Solomon can use that implements his social choice function in Nash Equilibrium? The answer is no.

**Definition (Maskin Monotonicity):** A social choice function $f$ satisfies Maskin Monotonicity if whenever the following two conditions are satisfied:

1. $f(\theta) = x$;
2. $L(x, \theta_i') \subset L(x, \theta_i)$ for all $i$;

then it also holds that $f(\theta') = x$. 

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In words, Maskin Monotonicity says that if \( x \) was chosen under the preference profile \( \theta \) and \( \theta' \) is such that \( x \) did not drop in the preference ranking of any agent, then \( x \) should be chosen under \( \theta' \) as well.

**Proposition:** A social choice function is strongly implementable in Nash equilibrium only if satisfies Maskin Monotonicity.

**Proof:** See class notes.

Not it is clear why King Solomon cannot implement his social choice function.

**Other Types of Implementation**

So far, we have assumed that the game played by the agents is a complete information game. In general, however, this might be a strong assumption. Indeed, firms don’t know each other’s cost, participants in an auction don’t know how much the other participants are bidding, etc. This motivates us to study mechanisms in incomplete information games. Before we do this, however, we need to introduce the standard equilibrium concept for games with incomplete information.

### 4.1 Bayesian Nash equilibrium

In a Bayesian game, each player has a utility function \( u_i(a, \theta_i) \), where \( \theta \in \Theta \) is a random variable chosen by nature according to a commonly known distribution \( F(\theta) \). Formally, a Bayesian game is

\[
\{I, \{A_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)\}.
\]

A strategy for each player \( i \) is a mapping \( a_i(\theta_i) \) from realizations of player \( i \)’s state to his actions.

**Definition (Bayesian Nash equilibrium):** A (pure strategy) Bayesian Nash equilibrium is a profile of decision rules \( (a_1(\cdot), ..., a_N(\cdot)) \) such that:

\[
E_{\theta}[u_i(a_i(\theta_1), a_{-i}(\theta_{-i}), \theta_i)] \geq E_{\theta}[u_i(a'_i(\theta_1), a_{-i}(\theta_{-i}), \theta_i)] \quad \forall i, \theta_i, a'_i
\]

Note that an equivalent condition is that for all \( i \) and \( \theta_i \) occurring with positive
probability

\[ E_{\theta_i} \left[ u_i(a_i(\theta_1), a_{-i}(\theta_{-i}), \theta_i) \mid \theta_i \right] \geq E_{\theta_i} \left[ u_i(a'_i(\theta_1), a_{-i}(\theta_{-i}), \theta_i) \mid \theta_i \right] \quad \forall i, \theta_i, a'_i \]

I.e., each player is maximizing his expected utility over types of the other players at each possible state of the world.

**Example:**

Consider the following game of incomplete information. With probability \( \mu \), the game played is the following:

<table>
<thead>
<tr>
<th></th>
<th>Silent</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silent</td>
<td>0, -2</td>
<td>-10, -1</td>
</tr>
<tr>
<td>Confess</td>
<td>-1, -10</td>
<td>-5, -5</td>
</tr>
</tbody>
</table>

This is a modification of the prisoner’s dilemma in which one of the prisoners is the cop’s brother. In this case, the cop can let his brother go if neither of the prisoner’s confesses.

With probability \( 1 - \mu \), the game played is the following:

<table>
<thead>
<tr>
<th></th>
<th>Silent</th>
<th>Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silent</td>
<td>0, -2</td>
<td>-10, -7</td>
</tr>
<tr>
<td>Confess</td>
<td>-1, -10</td>
<td>-5, -11</td>
</tr>
</tbody>
</table>

This is a game in which the second prisoner has high moral costs (equivalent to six years in jail) for confessing. The first prisoner’s preferences are the same as in the first game.

To find the Bayesian Nash equilibrium, note that the first prisoner has the same preferences in both games. His strategy will therefore not depend on the nature’s state. The second prisoner’s (strictly) dominant strategy is Confess with probability \( \mu \) and Silent with probability \( 1 - \mu \), and he will play it in equilibrium. Player 1 gets \(-10\mu\) from Silent and \(-(1 - \mu) - 5\mu\) from Confess. Therefore, his best response is to play Silent if \( \mu < 1/6 \) and Confess if \( \mu > 1/6 \).
4.2 Mechanism Design in Bayesian Games

We will start out with a few motivating examples. Before we do this, let’s focus attention on one desirable property of social choice functions:

**Definition (Ex-post efficiency):** A social choice function \( f : \Theta \rightarrow X \) is ex-post efficient if for no profile \( \theta \in \Theta \) there is an \( x \) s.t. \( u_i(x, \theta_i) \geq u_i(f(\theta), \theta_i) \) for all \( i \) with a strict inequality for some \( i \).

4.3 Single Unit Auction

There is a single good to be allocated among \( N \) agents. Monetary transfers can be made. An outcome is \((y_1, \ldots, y_N, t_1, \ldots, t_N)\), where each \( y_i \in \{0, 1\} \) (1 if the agents get the good and 0 otherwise), \( t_i \in \mathbb{R} \), and \( \sum_i t_i \leq 0 \) (you can burn money, but you can’t get transfer money from the outside).

The set of feasible outcomes is therefore:

\[
X = \{(y_1, \ldots, y_N, t_1, \ldots, t_N) : y_i \in \{0, 1\} \forall i, \quad \sum_i y_i = 1, \quad t_i \in \mathbb{R} \forall i, \quad \sum_i t_i \leq 0\}.
\]

Agents’ utility functions are given by:

\[
u_i(x, \theta_i) = y_i \theta_i + t_i,
\]

where each \( \theta_i \) is drawn independently from a uniform distribution of \([0, 1]\).

A social choice function is a mapping from \( \Theta \) to the set of feasible allocations.

In this environment, a social choice function is efficient if, for every \( \theta \), it gives the good to the highest agent and if \( \sum_i t_i(\theta) = 0 \).

Second Price Auction

Let’s first try a second price auction. There are two buyers and one seller. Each potential buyer submits a bit \( b_i \geq 0 \). The bids are then opened and the buyer with the highest bid gets the object. He then pays the second highest bid.
We studied the Bayesian game induced by this mechanism in a homework problem, although we didn’t know about Bayesian games yet. We showed then that it’s a weakly dominant strategy for all players to submit their $\theta_i$ as the bid.

The mechanism induced by the second price auction implements the social choice function below:

$$
\begin{align*}
  y_o(\theta) &= 0 \quad \forall \theta \\
  y_1(\theta) &= 1 \quad \text{if } \theta_1 \geq \theta_2 \\
  y_1(\theta) &= 0 \quad \text{if } \theta_1 < \theta_2 \\
  y_2(\theta) &= 1 \quad \text{if } \theta_2 \geq \theta_1 \\
  y_2(\theta) &= 0 \quad \text{if } \theta_2 < \theta_1 \\
  t_0(\theta) &= t_1(\theta) + t_2(\theta) \quad \forall \theta \\
  t_1(\theta) &= -\theta_2 y_1(\theta) \quad \forall \theta \\
  t_2(\theta) &= -\theta_1 y_2(\theta) \quad \forall \theta 
\end{align*}
$$

**First Price Auction**

Let’s now try a first price auction. Each potential buyer submits a bit $b_i \geq 0$. The bids are then opened and the buyer with the highest bid gets the object. He then pays his bid. Let’s consider equilibria in which each buyer’s strategy takes the form

$$
b_i(\theta_i) = \alpha_i \theta_i
$$

for some $\alpha_i \in [0, 1]$. For each $\theta_1$, buyer 1 wants to solve:

$$
\max_{b_1 \geq 0} (\theta_1 - b_1) \cdot \text{Prob}(b_2(\theta_2) \leq b_1)
$$

Since $b_2(\theta_2) = \alpha_2 \theta_2$, we can write the problem as

$$
\max_{b_1 \geq 0} (\theta_1 - b_1) \cdot \text{Prob}(\theta_2 \leq \frac{b_1}{\alpha_2})
$$

Clearly, buyer 1 would never bid above $\alpha_2$ (otherwise, he could lower his bid, leaving the probability unaffected, but increasing the earnings). Therefore, the problem becomes:

$$
\max_{b_1 \in [0, \alpha_2]} (\theta_1 - b_1) \cdot \text{Prob}(\theta_2 \leq \frac{b_1}{\alpha_2}).
$$
Finally, since \( \theta_2 \) is uniformly distributed, the problem becomes

\[
\max_{b_1 \in [0, \alpha_2]} (\theta_1 - b_1) \frac{b_1}{\alpha_2}.
\]

The solution to this problem is

\[
b_1(\theta_1) = \begin{cases} 
\frac{\theta_1}{2} & \text{if } \frac{\theta_1}{2} \leq \alpha_2 \\
\alpha_2 & \text{if } \frac{\theta_1}{2} > \alpha_2
\end{cases}
\]

Similarly, for buyer 2,

\[
b_2(\theta_2) = \begin{cases} 
\frac{\theta_2}{2} & \text{if } \frac{\theta_2}{2} \leq \alpha_1 \\
\alpha_1 & \text{if } \frac{\theta_2}{2} > \alpha_1
\end{cases}
\]

Letting \( \alpha_1 = \alpha_2 = 1/2 \), we see that \( b_i(\theta_i) = \frac{\theta_i}{2} \) is a Bayesian Nash equilibrium of this auction.

This Bayesian Nash equilibrium implements the social choice function below:

\[
y_o(\theta) = 0 \quad \forall \theta \\
y_1(\theta) = 1 \quad \text{if } \theta_1 \geq \theta_2 \\
y_1(\theta) = 0 \quad \text{if } \theta_1 < \theta_2 \\
y_2(\theta) = 1 \quad \text{if } \theta_2 \geq \theta_1 \\
y_2(\theta) = 0 \quad \text{if } \theta_2 < \theta_1 \\
t_0(\theta) = t_1(\theta) + t_2(\theta) \quad \forall \theta \\
t_1(\theta) = -\frac{1}{2} \theta_1 y_1(\theta) \quad \forall \theta \\
t_2(\theta) = -\frac{1}{2} \theta_2 y_2(\theta) \quad \forall \theta
\]

**Revenue Equivalence**

Notice that, at least in our examples of equilibria, the social choice functions implemented by first and second price auctions look quite different. They nevertheless generate the same revenue for the seller. You will be asked to show this in a homework problem. We will later study the general conditions under which revenue equivalence holds.
Surplus Extraction

Consider the social choice function \( f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta)) \) in which

\[
\begin{align*}
y_0(\theta) &= 0 \quad \forall \theta \\
y_1(\theta) &= 1 \quad \text{if} \quad \theta_1 \geq \theta_2 \\
y_1(\theta) &= 0 \quad \text{if} \quad \theta_1 < \theta_2 \\
y_2(\theta) &= 1 \quad \text{if} \quad \theta_2 \geq \theta_1 \\
y_2(\theta) &= 0 \quad \text{if} \quad \theta_2 < \theta_1 \\
t_0(\theta) &= t_1(\theta) + t_2(\theta) \quad \forall \theta \\
t_1(\theta) &= -\theta_1 y_1(\theta) \quad \forall \theta \\
t_2(\theta) &= -\theta_2 y_2(\theta) \quad \forall \theta
\end{align*}
\]

In words, the item goes to the buyer who values the item more, who pays his value to the seller (ties are broken arbitrarily). Notice that this social choice function is efficient, and also quite desirable from the seller’s point of view, since he gets to extract the entire output. Our question is: Can this social choice function above be implemented in Bayesian Nash Equilibrium?

Before we answer this more complicated question, let’s see if it’s an equilibrium for the buyers to report their types truthfully if they are paid according to the social choice function above. As we will see later, truthful implementation is closely connected to implementation in general.

Assume player 2 tells the truth. Then, given any \( \theta_1 \), player 1 chooses a report \( \hat{\theta}_1 \) to solve

\[
\max_{[0, \hat{\theta}_1]} \theta_1 - \hat{\theta}_1) Pr(\hat{\theta}_1 \geq \theta_2)
\]

or

\[
\max_{[0, \hat{\theta}_1]} (\theta_1 - \hat{\theta}_1)(\hat{\theta}_1)
\]

The first order conditions are

\[
\theta_1 = 2\hat{\theta}_1
\]

or

\[
\hat{\theta}_1 = \frac{\theta_1}{2}.
\]

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Thus, player 1 always has incentives to under-report his true state. While this decreases the probability of getting the object, the cost is worth it because there is also a benefit of paying less. The social choice function cannot be implemented truthfully.

4.4 Bayesian Implementation

Now assume that players have preferences $u_i(x, \theta_i)$ over the set of outcomes $X$ that are indexed by $\theta_i$. Notice that any mechanism $\Gamma = (M, g)$ induces a Bayesian game. We can define an equilibrium for this game as follows.

**Definition:** A strategy profile $(m_1(\cdot), ..., m_N(\cdot))$ is a Bayesian Nash equilibrium of $\Gamma$ if

$$E_{\theta^i} [u_i(g(m_i(\theta_i), m_{-i}(\theta_i)), \theta_i)|\theta_i)] \geq E_{\theta^i} [u_i(g(\hat{m}_i(\theta_i), m_{-i}(\theta_i)), \theta_i)|\theta_i)] \quad \forall i, \theta_i, \hat{m}_i(\cdot).$$

**Definition (Bayesian Implementation):** We say that the mechanism $\Gamma = (M, g)$ implements the social choice function $f$ in Bayesian Nash equilibrium if there exists a strategy profile $(m_1(\cdot), ..., m_N(\cdot))$ satisfying the following conditions:

- It is a Bayesian Nash equilibrium of $\Gamma$;
- $f(\theta) = g(m(\theta)).$

To study truthful implementation, we need to define a direct mechanism. This is a mechanism in which $\Theta = M$ and $g = f$.

**Definition (Truthful Implementation):** We say that that the social choice function $f$ is **truthfully implementable in Bayesian Nash equilibrium** (or **Bayesian incentives compatible**) if telling the truth is a Bayesian Nash equilibrium of the direct mechanism for any $\theta$. That is,

$$E_{\theta^i} [u_i(f(\theta_i, \theta_i), \theta_i)|\theta_i)] \geq E_{\theta^i} [u_i(f(\hat{\theta}_i, \theta_i), \theta_i)|\theta_i)] \quad \forall i, \theta_i, \hat{\theta}_i.$$

We can now state one of the most important results in the mechanism design literature.
**Proposition (Revelation Principle):** Suppose that a social choice function $f$ is implementable in Bayesian Nash Equilibrium. Then $f$ is truthfully implementable in Bayesian Nash Equilibrium.

The proof of the revelation principle couldn’t be simpler; it’s given below.

**Proof:**

Let $\Gamma = (M, g)$ be the mechanism that implements $f$ in Bayesian Nash equilibrium. Then,

$$E_{\theta_i} [u_i(g(m_i(\theta_i), m_{-i}(\theta_i)), \theta_i)] \geq E_{\theta_i} [u_i(g(\hat{m}_i(\theta_i), m_{-i}(\theta_i)), \theta_i)] \quad \forall i, \theta_i, \hat{m}_i(\cdot).$$

Take any $\hat{\theta}_i$ and let $\hat{m}_i(\theta_i) = m_i(\theta_i)$. Then,

$$E_{\theta_i} [u_i(g(m_i(\theta_i), m_{-i}(\theta_i)), \theta_i)] \geq E_{\theta_i} [u_i(g(\hat{m}_i(\hat{\theta}_i), m_{-i}(\theta_i)), \theta_i)] \quad \forall i, \theta_i, \hat{m}_i(\cdot).$$

By definition of Bayesian Nash implementation, $g(m(\theta)) = f(\theta)$, so

$$E_{\theta_i} [u_i(f(\theta), \theta_i)] \geq E_{\theta_i} [u_i(f(\hat{\theta}_i, \theta_{-i}), \theta_i)] \quad \forall i, \theta_i, \hat{m}_i(\cdot).$$

This is precisely the condition for being truthfully implementable. **QED**

We now see that there is no auction that can implement our favorite social choice function that maximizes the seller’s revenue. It is not truthfully implementable and hence not implementable in Bayesian Nash equilibrium.

### 4.5 Revenue equivalence

Our final result in the study of mechanism design is one of the most important applications to auctions.

**Theorem (Revenue equivalence theorem):** Suppose in our single unit auction set-up that a given pair of Bayesian Nash equilibria of two different auction procedures are such that:

- Each buyer has the same probability of getting the good in both auctions.
- Each buyer has the same expected utility in both auctions if his type is 0 (or, more generally, the lowest possible one: $\hat{\theta}_i$).
Then these equilibria generate the same expected revenue for the seller.

**Proof:** Define

\[ \bar{y}_i(\hat{\theta}_i) = E_{\theta_{-i}}[y_i(\hat{\theta}_i, \theta_{-i})] \]
\[ \bar{t}_i(\hat{\theta}_i) = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i})] \]
\[ U_i(\theta_i) = \theta_i \bar{y}_i(\theta_i) + \bar{t}_i(\theta_i) \]

We will make use of the following characterization from MWG (see Proposition 23.D.2):

\[ U_i(\theta_i) = U_i(\theta_i) + \int_{\theta_i}^{\bar{\theta}_i} \bar{y}_i(s)ds. \]

Now, the seller’s expected revenue from any buyer \( i \) is:

\[ E(-t_i(\theta_i)) = E_{\theta_{-i}}(-\bar{t}_i(\theta_i)) \]
\[ = \int_{\theta_i}^{\bar{\theta}_i} [\theta_i \bar{y}_i(\theta_i) - U_i(\theta_i)] \phi_i(\theta_i) d\theta_i \]
\[ = \int_{\theta_i}^{\bar{\theta}_i} \left[ \theta_i \bar{y}_i(\theta_i) - \int_{\theta_i}^{\bar{\theta}_i} \bar{y}_i(s)ds \right] \phi_i(\theta_i) d\theta_i - U_i(\theta_i) \]

By integration by parts,

\[ \int_{\theta_i}^{\bar{\theta}_i} \left[ \int_{\theta_i}^{\bar{\theta}_i} \bar{y}_i(s)ds \right] \phi_i(\theta_i) d\theta_i = \int_{\theta_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i) d\theta_i - \int_{\theta_i}^{\bar{\theta}_i} \Phi_i(\theta_i) \bar{y}_i(\theta_i) d\theta_i. \]

Therefore,

\[ E(-t_i(\theta_i)) = \int_{\theta_i}^{\bar{\theta}_i} \bar{y}_i(\theta_i) \left( \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \phi_i(\theta_i) d\theta_i - U_i(\theta_i) \]
\[ = \int_{\theta_i}^{\bar{\theta}_i} \prod_{j \neq i} y_j(\theta_j) \left( \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \prod_{j \neq i} \phi_j(\theta_j) d\theta_1...d\theta_N - U_i(\theta_i). \]

Summing over all players, the seller’s revenue is:

\[ \sum_i \left\{ \int_{\theta_i}^{\bar{\theta}_i} \prod_{j \neq i} y_j(\theta_j) \left( \theta_i - \frac{1 - \Phi_i(\theta_i)}{\phi_i(\theta_i)} \right) \prod_{j \neq i} \phi_j(\theta_j) d\theta_1...d\theta_N \right\} - \sum_i U_i(\theta_i) \]

Clearly, the expression only depends on the sets of \( y_i(\theta) \) and \( U_i(\theta_i) \).