1 Normal Form Games

A normal form game is $(I, (A^i)_{i=1,...,n}, (u^i)_{i=1,...,n})$, where $\forall i$ A_i is an action set, $A = \times_{i=1}^n A^i$, and $u_i : A \to \mathbb{R}$ $\forall i$.

$I = \{1, ..., n\}$ is the set of players.

Assume A^i is finite for all i.

Examples: Coordination, Matching pennies, Prisoner's dilemma, Battle of the Sexes.

1.1 Dominance

Let $S^i = \Delta(A^i) = \{(s(a_1^i), ..., s(a_{k_i}^i)) : \forall i,\ s(a_i) \geq 0, \sum_{A^i} s(a^i) = 1\}$.

A mixed extension of a normal form game is $(I, (S^i)_{i=1,...,n}, (u^i)_{i=1,...,n})$, where $\forall$ $S^i = \Delta(A^i)$, $S = \prod_{i=1}^n S^i$ and $u^i : S \to \mathbb{R}$ is defined by

$$u^i(s^1, ..., s^n) = \sum_{a \in A} u^i(a) \prod_{i=1}^n s^i(a^i).$$

We write $Pr_s(a) = \prod_{i=1}^n s^i(a^i) \in \Delta A$.

Example: Show that in the game below, the player can get a better payoff by mixing T and M than by playing B, no matter what his belief is about what his partner is doing.

	L	R
T	3	0
M	0	3
B	1	1

We say $s^i \in S^i$ strictly dominates $a^i \in A^i$ iff for all a^{-i}

$$u^i(s^i, a^{-i}) > u^i(a^i, a^{-i}).$$

Alternatively,

$$s^i D_2 a^i \Leftrightarrow \forall s^{-i} \in S^{-i} \quad u^i(s^i, s^{-i}) > u_i(a^i, s^{-i})$$

or

$$s^i D_3 a^i \Leftrightarrow \forall \mu \in \Delta(A^{-i}) \quad u^i(s^i, \mu) > u_i(a^i, \mu)$$

Exercise: $s^i D_3 a^i \Leftrightarrow s^i D_2 a^i \Leftrightarrow s^i D_1 a^i$.

Example: Note that T and L are both dominated in the game below.

	L	R
T	-2,-2	-10,-1
B	-1,-10	-5,-5

This leads to the counter-intuitive prediction of playing (B,R). Of course this doesn't happen in real life.

Example:

	L	R
T	3	0
M	0	3
B	x	x

Consider a belief p for Player 1 that Player 2 chooses L. Note that if $x < \frac{3}{2}$, B is never a best response. For every belief, Player 1 is better of playing T or M. Dually, $\exists s^1 \in S^1$ that dominates B.

If $x = \frac{3}{2}$, there exists a belief ($p = 0.5$) for which B is a best response. Dually, B is not strictly dominated.

This example suggests that an action is never a best response if and only if it is strictly dominated by a strategy.

Definition: An action $a^i \in A^i$ is never a best response if there is no $\mu \in \Delta(A^{-i})$ such that $u^i(a^i, \mu) \geq u^i(b^i, \mu)$ for all b^i.

Theorem: An action $a^i \in A^i$ is strictly dominated if and only if it is never a best response.

One direction is easy to prove (see your class notes). The proof for the other direction can be found in Osborne and Rubinstein.

1.2 IESDA

We illustrate this with examples:

	L	R
T	0,-2	-10,-1
B	-1,-10	-5,-5

	L	R
T	3,0	0,1
M	0,0	3,1
B	1,1	1,0

Example (Cournot Duopoly):

Consider a two player game with two firms $i = 1, 2$. Each firm faces the demand curve $p = a - b(q_1 + q_2)$ and per-unit costs of production c. Show that iterated elimination of strictly dominated actions yields a unique outcome in which each firm produces $\frac{a-c}{3b}$.

1.3 Rationalizability

An action $a^i \in A^i$ is rationalizable if you can construct an infinite chain of beliefs (using other rationalizable actions) to justify playing it. Unlike in Nash Equilibrium, the beliefs do not have to be correct. In the examples below, we use R^i to denote the set of rationalizable actions of player i.

Example:

	L	R
T	3,0	0,1
M	0,0	3,1
B	1,1	1,0

$(R^1, R^2) = (\{M\}, \{R\})$.

Example 2:

	L	R
T	3,1	0,0
B	0,0	1,3

$(R^1, R^2) = (\{T,\ B\}, \{L,\ R\})$.

As we discussed before, $D \Leftrightarrow NBR$. D is related to iterated dominance, while NBR is related to rationalizability. I turns out that the set of actions that survives *complete* IESDA is unique and equal to the set of all rationalizable actions (we skip the proof in this class).

Example:

	b_1	b_2	b_3	b_4
a_1	0,7	2,5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7,0	2,5	0,7	0,1
a_4	0,0	0,-2	0,0	10,-1

It's easy to show that the set of all rationalizable actions is $(R^1, R^2) = (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$ (eliminate b_4 in step 1, and a_4 in step 2).

1.4 Weak Dominance

Just because this is used in one of the papers we will discuss, here is a different notion of dominance:

Definition: $s^i W a^i \quad \Leftrightarrow$

- For all $a^{-i} \in A^{-i} \quad u^i(s_i, a^{-i}) \geq u^i(a_i, a^{-i})$.
- There exists $b^{-i} \in A^{-i}$ such that $u^i(s_i, b^{-i}) > u^i(a_i, b^{-i})$.

1.5 Cognitive Hierarchy

We illustrate level-k and cognitive hierarchy models with an example. Consider the following "traveler's dilemma" game:

- $I = \{1, 2\}$
- $A^i = \{180, 181, ..., 300\}$
- $u^i(a^i, a^{-i}) = \begin{cases} a^i + R & \text{if } a^i < a^{-i} \\ a^{-i} - R & \text{if } a^i > a^{-i}, \\ a^{-i} & \text{if } a^i = a^{-i} \end{cases} \qquad \text{where } R > 1.$

1.5.1 Rationality in a Traveler's Dilemma

A decision maker is rational if she best-responds to her beliefs. In a traveler's dilemma game, any $a^i < 300$ can be a rational response to some belief. For example, if the other player chooses 300, your best response is to choose 299. The choice of 300, on the other hand, is never a best response and therefore irrational. Thus, the rational choice prediction in a traveler's dilemma game is that no one chooses an action of 300. Notice that this prediction is weak because many different choices are consistent with rationality.

1.5.2 Rationalizability in a Traveler's Dilemma

To find the rationalizable actions, we use iterated elimination of strictly dominated actions. In step 1, we eliminate 300, since it is never a best response and therefore strictly dominated. In step 2, we eliminate 299, and so on, until all we are left with is the lower bound of 180. Thus, the traveler's dilemma has a unique rationalizable action: 180. All other action choices are putting positive probabilities on other players choosing actions that are not rationalizable.

1.5.3 Level-k

A Level-k model decomposes associates rational choices with player types as follows. A Level-0 player is assumed to choose a random action between 180 and 300. Note that a Level-0 player will choose $a_0^i = 240$ on average. For $k > 0$, a Level-k player best-responds to a Level-(k-1) player by choosing $a_k^i = a_{k-1}^i - 1$.

1.5.4 Cognitive hierarchy

A cognitive hierarchy model, is the same as a Level-k model for $k < 2$. For $k \geq 2$, the player is assumed to best respond to some pre-specified distribution of types of players below him. I.e., this model relaxes the assumption that a Level-k player puts probability 1 on facing a Level-k-1 player. For example, a Level-2 player can best-respond to a 50/50 distribution of Level-0 and Level-1 players.

1.5.5 Beauty Contest

Another game that researchers often use to study hierarchical thinking is the p-beauty contest. In a p-beauty contest, $A^i = [0, \bar{a}]$, and the players are incentivized go guess some fraction $p \in (0, 1)$ of the average $\frac{1}{N} \sum a^i$. In a Level-k model, a Level-0 player is assumed to choose randomly (notice that this makes his or her expected response $\bar{a}/2$). A Level-1 player responds by choosing $p\bar{a}/2$, a Level-2 player responds by choosing $p^2\bar{a}/2$, and so on.