1 Normal Form Games

A normal form game is $(I, (A^i)_{i=1,\dots,n}, (u^i)_{i=1,\dots,n})$, where $\forall i$ A_i is an action set, $A = \times_{i=1}^n A^i$, and $u_i : A \to \mathbb{R}$ $\forall i$.

$I = \{1, ..., n\}$ is the set of players.

Assume A^i is finite for all i.

Examples: Coordination, Matching pennies, Prisoner's dilemma, Battle of the Sexes.

1.1 Dominance

Let $S^i = \Delta(A^i) = \{(s(a_1^i), ..., s(a_{k_i}^i)) : \forall i,\ s(a_i) \geq 0, \sum_{A^i} s(a^i) = 1\}$.

A mixed extension of a normal form game is $(I, (S^i)_{i=1,\dots,n}, (u^i)_{i=1,\dots,n})$, where $\forall$ $S^i = \Delta(A^i)$, $S = \prod_{i=1}^n S^i$ and $u^i : S \to \mathbb{R}$ is defined by

$$u^i(s^1, ..., s^n) = \sum_{a \in A} u^i(a) \prod_{i=1}^n s^i(a^i).$$

We write $Pr_s(a) = \prod_{i=1}^n s^i(a^i) \in \Delta A$.

Example: Show that in the game below, the player can get a better payoff by mixing T and M than by playing B, no matter what his belief is about what his partner is doing.

	L	R
T	3	0
M	0	3
B	1	1

We say $s^i \in S^i$ strictly dominates $a^i \in A^i$ iff for all a^{-i}

$$u^i(s^i, a^{-i}) > u^i(a^i, a^{-i}).$$

Alternatively,

$$s^i D_2 a^i \Leftrightarrow \forall s^{-i} \in S^{-i} \quad u^i(s^i, s^{-i}) > u_i(a^i, s^{-i})$$

or

$$s^i D_3 a^i \Leftrightarrow \forall \mu \in \Delta(A^{-i}) \quad u^i(s^i, \mu) > u_i(a^i, \mu)$$

Exercise: $s^i D_3 a^i \Leftrightarrow s^i D_2 a^i \Leftrightarrow s^i D_1 a^i$.

Example: Note that T and L are both dominated in the game below.

	L	R
T	-2,-2	-10,-1
B	-1,-10	-5,-5

This leads to the counter-intuitive prediction of playing (B,R). Of course this doesn't happen in real life.

Example:

	L	R
T	3	0
M	0	3
B	x	x

Consider a belief p for Player 1 that Player 2 chooses L. Note that if $x < \frac{3}{2}$, B is never a best response. For every belief, Player 1 is better of playing T or M. Dually, $\exists s^1 \in S^1$ that dominates B.

If $x = \frac{3}{2}$, there exists a belief ($p = 0.5$) for which B is a best response. Dually, B is not strictly dominated.

This example suggests that an action is never a best response if and only if it is strictly dominated by a strategy.

Definition: An action $a^i \in A^i$ is never a best response if there is no $\mu \in \Delta(A^{-i})$ such that $u^i(a^i, \mu) \geq u^i(b^i, \mu)$ for all b^i.

Theorem: An action $a^i \in A^i$ is strictly dominated if and only if it is never a best response.

One direction is easy to prove (see your class notes). The proof for the other direction can be found in Osborne and Rubinstein.

1.2 IESDA

We illustrate this with examples:

	L	R
T	0,-2	-10,-1
B	-1,-10	-5,-5

	L	R
T	3,0	0,1
M	0,0	3,1
B	1,1	1,0

Example (Cournot Duopoly):

Consider a two player game with two firms $i = 1, 2$. Each firm faces the demand curve $p = a - b(q_1 + q_2)$ and per-unit costs of production c. Show that iterated elimination of strictly dominated actions yields a unique outcome in which each firm produces $\frac{a-c}{3b}$.

1.3 Rationalizability

An action $a^i \in A^i$ is rationalizable if you can construct an infinite chain of beliefs (using other rationalizable actions) to justify playing it. Unlike in Nash Equilibrium, the beliefs do not have to be correct. In the examples below, we use R^i to denote the set of rationalizable actions of player i.

Example:

	L	R
T	3,0	0,1
M	0,0	3,1
B	1,1	1,0

$(R^1, R^2) = (\{M\}, \{R\})$.

Example 2:

	L	R
T	3,1	0,0
B	0,0	1,3

$(R^1, R^2) = (\{T,\ B\}, \{L,\ R\})$.

As we discussed before, $D \Leftrightarrow NBR$. D is related to iterated dominance, while NBR is related to rationalizability. I turns out that the set of actions that survives *complete* IESDA is unique and equal to the set of all rationalizable actions (we skip the proof in this class).

Example:

	b_1	b_2	b_3	b_4
a_1	0,7	2,5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7,0	2,5	0,7	0,1
a_4	0,0	0,-2	0,0	10,-1

It's easy to show that the set of all rationalizable actions is $(R^1, R^2) = (\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\})$ (eliminate b_4 in step 1, and a_4 in step 2).

1.4 IEWDA

Definition: $s^i W a^i \quad \Leftrightarrow$

- For all $a^{-i} \in A^{-i} \quad u^i(s_i, a^{-i}) \geq u^i(a_i, a^{-i})$.
- There exists $b^{-i} \in A^{-i}$ such that $u^i(s_i, b^{-i}) > u^i(a_i, b^{-i})$.

Example: Using the game below, show that the set of actions surviving IEWDA is not unique.

	L	R
T	1,1	0,0
M	1,1	2,1
B	0,0	2,1

Example (Vickrey auction): An object is auctioned off to N bidders. Bidder i's valuation of the object (in monetary terms) is v_i. The auction rules are that each player submit a bid (a nonnegative number) in a sealed envelope. The envelopes are then opened, and the bidder who has submitted the highest bid gets the object but pays the auctioneer the amount of the *second-highest* bid. If more than one bidder submits the highest bid, each gets the object with equal probability. Show that submitting a bid of v_i with certainty is a weakly dominant strategy for bidder i.

1.5 Nash Equilibrium

Definition (Pure Best Reply): $PBR^i(s) = \{a^i \in A^i : u^i(a^i, s^{-i}) \geq u^i(b^i, s^{-i}) \quad \forall b^i \in A^i\}$.

Note that this set is nonempty for a finite game.

Definition (Best Reply): $BR^i(s) = \{s^i \in S^i : u^i(s^i, s^{-i}) \geq u^i(b^i, s^{-i}) \quad \forall b^i \in A^i\}$.

Example: Find $PBR^i(s)$ and $BR^i(s)$ in the following game:

	L	R
T	3,1	0,0
B	0,0	1,3

Definition:

$$BR(s) = \times_{i=1}^{N} BR^i(s).$$

Definition (Nash Equilibrium): A Nash Equilibrium of a NFG is a strategy profile $\hat{s}$ s.t. $\hat{s} \in BR(\hat{s})$.

Nash's Theorem: Let $G = (I, (A^i)_{i=1,\ldots,n}, (u^i)_{i=1,\ldots,n})$ be a (finite) normal form game. The set of Nash Equilibrium strategy profiles in this game is nonempty.

Philosophical point: Keep in mind that Nash equilibrium is *not* an implication of rationality. Rationalizability is an implication of rationality. Nash has stronger "epistemic" assumptions (assumptions about knowledge). E.g., it assumes that every player knows what every other player is playing.

2 Dynamic Games

An extensive form (dynamic) game has the following basic ingredients: set of players I (possibly including nature), set of nodes X, set of final nodes Z, and set of utility functions $u^i : Z \to \mathbb{R}$. If nature controls the initial node, we denote this node by o. Any description of an EFG must also specify:

- How the nodes are ordered;
- How the nodes are allocated among players, including nature;
- How nature determines its moves;
- What information the players have;
- What moves the player can make at every node.

What follows is a more formal description of these elements than what you need to know for the exam. It is included in these notes for completion, but all you really need to have is an intuitive understanding of nodes, information sets, and player's choices.

Ordering of nodes

Formally, we require $\succsim$ a partial order over the set of nodes ($x \succsim y$ means x comes before y) with the following properties:

1. $\succsim$ is reflexive, antisymmetric, transitive;
2. $\forall x \quad 0 \succsim x$;
3. $\forall x \in X$, $p(x)$=the set of predecessors of x=$\{y : y \succsim x\}$ is linearly ordered by $\succsim$.

Players' partition

A players' partition is partition $(P^1, P^2, ..., P^N)$ of the set of non-terminal nodes:

1. $\forall i \neq j,\ P^i \cap P^j = \emptyset$;
2. $\cup_i P^i = X \setminus \{o, Z\}$.

Nature's move, if nature is a player

The probability of nature's move is a probability distribution $p_o \in \Delta(S(o))$, where $S(o)$ is the set of immediate successors of the initial node.

Information partition

For every player i, the information partition $U^i = \{u, v, w, ...\}$ is a partition of P^i. That is,

1. $u \cup v \cup w \cup ... = P^i$;
2. $u \neq v \Rightarrow u \cap v = \emptyset$.

Players' choices

Let $S(x)$ the set of immediate successors of node x. For any information set u, let $\cup_{x \in u} S(x)$ to be the set of immediate successors of u. For all $i, u \in U^i$, C_u is the set of choices i can make at u. This is a partition of $\cup_{x \in u} S(x)$ such that each set contains one and only one of the elements of $S(x)$ for every $x \in u$.

Additional properties

We might want to impose some additional intuitive properties on extensive form games. In particular, we might want to rule out situations where a player doesn't remember if he moved or not, and situtations where the player has different numbers of actions at different nodes in the same information set.

Formally, we might assume that the game is **linear**. This means that $\forall i$, $\forall u \in U^i$, $\forall z \in Z$, $\quad \#\{p(z) \cap u\} \leq 1$. In words, the path of predecessors of any terminal node intersects any information set at most once. This rules out situations where players forget whether or not they moved.

We might also assume that $\forall i,\ \forall u \in U^i,\ \forall x_1, x_2 \in u, \quad \#S(x_1) = \#S(x_2)$. This rules out situations where different nodes in the same information set have different numbers of actions. This would be nonintuitive since we expect that a player would be able to distinguish between such nodes.

Finally, we might want to assume that players remember what they did. Formally, a game satisfies **perfect recall** if there exists no player i with information sets u and v and a choice $c \in C_u$ such that $x_1, x_2 \in v$, x_1 follows c but x_2 does not follow c.

2.1 Strategies in Extensive Form Games

Definition (Pure Strategies): A pure strategy specifies a choice at every information set. Formally, the set of pure strategies $A^i = \times_{u \in U^i} C_u = \{(a^i_u)_{u \in U^i} : a^i_u \in C_u \quad \forall u\}$.

Definition (Mixed Strategies): A mixed strategy is a probability distribution on A^i. Formally, $S^i = \Delta(A^i)$.

2.2 Normal Representation

Every extensive form game has a normal form representation, with the same set of players I and $A^i = \times_{u \in U^i} C_u$, as defined above. Every action profile leads to a terminal node, and $u^i(z)$ at that node gives player i's utility. If nature is one of the players, then for every pure strategy profile $(a^1, ..., a^N)$ of the players, there is a unique probability on the final nodes given by $Pr_{p_0,a} \in \Delta(Z)$ and hence an expected utility vector. Thus, in general, the NFG representation of an EFG has:

- $I = \{1, ..., N\}$;
- For every i, A^i the set of pure strategies;
- $u^i(a) = E_{Pr_{p_0,a}} u^i(\cdot)$.

Note: We say that two pure strategies b^i and c^i are *payoff equivalent* if $u^j(b^i, a^{-i}, d^i) =$

$u^j(c^i, a^{-i})\ \forall a^{-i} \in A^{-i}, \quad \forall j \in I$. I.e., no player cares if i uses b^i or c^i. We can replace each set of payoff-equivalent strategies with a single strategy and relabel. The result is called the (purely) reduced NFG representation.

Example: See class notes.

2.3 NE of EFG

Definition (NE of EFG): The NE of the EFG is the NE of the mixed extension of the associated NFG.

Example: See class notes.

Theorem: The set of NE of any EFG is non-empty.

Proof: Any EFG has a NFG representation. We showed that the set of NE of any NFG is non-empty earlier.