Homework 4

1. Consider the chicken game:

	L	R
T	6,6	2,7
B	7,2	0,0

Show that $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ is a correlated equilibrium of this game.

Answer: See class notes.

2. Let $s \in S$ be a Nash Equilibrium profile. Show formally that $\prod_{i=1}^N s^i \in \Delta(A)$ is a correlated equilibrium.

Answer: By definition of Nash Equilibrium,

$$u^i(s^i, s^{-i}) \geq u^i(\hat{s}^i, s^{-i}) \quad \forall i, \quad \forall \hat{s}^i \in S^i.$$

In particular,

$$u^i(s^i, s^{-i}) \geq u^i(b^i, s^{-i}) \quad \forall i, \quad \forall b^i \in A^i.$$

Recall that for all a^i with $s^i(a^i) > 0$, $u^i(a^i, s^{-i}) = u^i(s^i, s^{-i})$. Thus, for all a^i with $s^i(a^i) > 0$

$$u^i(a^i, s^{-i}) \geq u^i(b^i, s^{-i}) \quad \forall i, \quad \forall b^i \in A^i.$$

Let's expand the utilities:

$$\sum_{a^{-i} \in A^{-i}} u^i(a^i, a^{-i})[\prod_{j \neq i} s^j(a^j)]) \geq \sum_{a^{-i} \in A^{-i}} u^i(b^i, a^{-i})[\prod_{j \neq i} s^j(a^j)]$$

Multiply both sides by $s^i(a^i)$ to get:

$$\sum_{a^{-i} \in A^{-i}} u^i(a^i, a^{-i})[\prod_{j} s^j(a^j)]) \geq \sum_{a^{-i} \in A^{-i}} u^i(b^i, a^{-i})[\prod_{j} s^j(a^j)]$$

Let $\mu = \prod_{i=1}^N s^i$. Then, another way of writing the inequality above is:

$$\sum_{a^{-i} \in A^{-i}} u^i(a^i, a^{-i})\mu(a) \geq \sum_{a^{-i} \in A^{-i}} u^i(b^i, a^{-i})\mu(a).$$

Since this is true for all i, for all a^i with $\mu(a^i) > 0$, and for all b^i, μ is a correlated equilibrium.

3. Show that every convex combination of correlated equilibrium payoff profiles is a correlated equilibrium payoff profile.

Answer: Let $\mu_1 \in \Delta(A)$ and $\mu_2 \in \Delta(A)$ be correlated equilibria. It suffices to show that for any $\lambda \in (0,1)$, $\lambda\mu_1 + (1-\lambda)\mu_2$ is a correlated equilibrium, as well. By definition of correlated equilibrium,

$$\sum_{a^{-i}} u^i(a^i, a^{-i})\mu_1(a^{-i}|a^i) \geq \sum_{a^{-i}} u^i(b^i, a^{-i})\mu_1(a^{-i}|a^i).$$

and

$$\sum_{a^{-i}} u^i(a^i, a^{-i})\mu_2(a^{-i}|a^i) \geq \sum_{a^{-i}} u^i(b^i, a^{-i})\mu_2(a^{-i}|a^i).$$

Multiply both sides of the inequality above by λ and both sides of the inequality below by $(1-\lambda)$, and then add the two left-hand sides and the two-right hand sides.

4. Show that every action used with positive probability in a correlated equilibrium is rationalizable.

Answer: Let a^i be an action played with positive probability in some correlated equilibrium $\hat{\mu}$. Let $R = (R^1, ..., R^2)$, where $R^j = \{a^j \in A^j \ : \ \hat{\mu}(a^j) > 0\}$. We now need to check that the definition of rationalizability is satisfied.

(1) Clearly, $a^i \in R^i$.

(2) Clearly, $R^j \subset A^j$ for all j.

(3) For all j and for all $b^j \in R^j$, consider the distribution $\hat{\mu}(\cdot|b^j)$. By definition of R, this distribution has support on R^{-j}. By definition of correlated equilibrium, b^j is a best response to $\hat{\mu}(\cdot|b^j)$.

Homework 5

4. Finish verifying that the conditions of Proposition 2 in Section 1.10 of our notes are satisfied for the example we looked at in class.

Answer: The state space and the payoffs are given below (we are assuming that the payoffs are common knowledge):

	L	R
U	$2,3,0$	$2,0,0$
D	$0,3,0$	$0,0,0$

A

	L	R
U	$0,0,0$	$0,2,0$
D	$3,0,0$	$3,2,0$

B

State	α	β	γ	$\boldsymbol{\delta}$	ϵ	ξ
Probability×63	32	16	8	**4**	2	1
1's action	U	D	D	$\boldsymbol{D}$	D	D
2's action	L	L	L	$\boldsymbol{L}$	L	L
3's action	A	B	A	$\boldsymbol{B}$	A	B
1's partition	$\{\alpha\}$	$\{\beta$	$\gamma\}$	$\{\boldsymbol{\delta}$	$\epsilon\}$	$\{\xi\}$
2's partition	$\{\alpha$	$\beta\}$	$\{\gamma$	$\boldsymbol{\delta}\}$	$\{\epsilon$	$\xi\}$
3's partition	$\{\alpha\}$	$\{\beta\}$	$\{\gamma\}$	$\{\boldsymbol{\delta}\}$	$\{\epsilon\}$	$\{\xi\}$

Consider the state δ.

Let's check that the players know each other's beliefs. Recall that i knows j's belief at ω if $P^i(\omega) \subset \{\omega' : \mu^j(\omega) = \mu^j(\omega')\}$. Let's start with player 1. $P^1(\delta) = \{\delta, \epsilon\}$. Since $\mu^2(\delta)(DB) = \mu^2(\epsilon)(DB) = \frac{1}{3}$ and $\mu^2(\delta)(DA) = \mu^2(\epsilon)(DA) = \frac{2}{3}$, it follows that player 1 knows player 2's belief. Since $\mu^3(\delta)(DL) = \mu^3(\epsilon)(DL) = 1$, it follows that player 1 knows player 3's belief. Hence, player 1 knows the beliefs of both other players. Now let's look at player 2. $P^2(\delta) = \{\gamma, \delta\}$. Since $\mu^1(\delta)(LB) = \mu^1(\gamma)(LB) = \frac{2}{3}$ and $\mu^1(\delta)(LA) = \mu^1(\gamma)(LA) = \frac{1}{3}$, player 2 knows player 1's belief. Since $\mu^3(\gamma)(LD) = \mu^3(\delta)(LD) = 1$, player 2 knows player 3's belief. Hence, player 2 knows the beliefs of both of the other players. Now let's look at player 3. $P^3(\delta) = \{\delta\}$. Since player 3 knows the state, he clearly knows the other players' beliefs at this state. Therefore, all of the players know the other players' beliefs.

We now need to check that players' beliefs are consistent with their knowledge. For player i, this means that the support of $\mu^i(\omega)$ is a subset of $\{a^{-i} \in A^{-i} : \sigma^{-i}(\omega') = a^{-i}$ for some $\omega' \in P^i(\omega)\}$. Let's start with player 1. The support of $\mu^1(\delta)$ is $\{LB, LA\}$. $LA = \sigma^{-1}(\delta)$ and $LB = \sigma^{-1}(\epsilon)$. Since $P^1(\delta) = \{\delta, \epsilon\}$, player 1's beliefs are consistent with his knowledge. Now let's look at player 2. The support of $\mu^2(\delta)$ is $\{DA, DB\}$. $DA = \sigma^{-2}(\gamma)$ and $DB = \sigma^{-2}(\delta)$. Since $P^2(\delta) = \{\gamma, \delta\}$, player 2's beliefs are consistent with his knowledge. Finally, let's look at player 3. The support of $\mu^3(\delta)$ is $\{DL\}$. Since $DL = \sigma^{-3}(\delta)$, player 3's beliefs are consistent with his knowledge. Therefore, all of the players have beliefs consistent with their knowledge.

Finally, let's check that all of the players know that the other players are rational. Let's start with player 1. $P^1(\delta) = \{\delta, \epsilon\}$. Both at δ and ϵ, player 2 plays L against the belief that the other players are playing DB with probability $\frac{1}{3}$ and DA with probability $\frac{2}{3}$. This is rational. Hence, player 1 knows that player 2 is rational. Player 1 also knows that player 3 is rational since anything is rational for player 3. Hence, player 1 knows that the other players are rational. Now let's look at player 2. $P^2(\delta) = \{\gamma, \delta\}$. Both at γ and δ, player 1 is playing D against the belief that the other players are playing LA with probability $\frac{1}{3}$ and LB with probability $\frac{2}{3}$. This is rational. Player 2 also knows that player 3 is rational since anything is rational for player 3. Hence, player 2 knows that the other players are rational. $P^2(\delta) = \{\delta\}$. As already discussed, both player 1 and player 2 are rational at δ. Hence, player 3 knows the other players to be rational.

It follows that the conditions of the proposition are satisfied at δ. Nevertheless, the players' conjectures do not form a Nash equilibrium. Indeed, they do not even agree.

Homework 6

Question 1 (Knowledge)

Let $I = \{1,2\}$, $A^1 = \{U, D\}$, $A^2 = \{L, R\}$. The utilities are given below and assumed to be common knowledge:

	L	R
U	2,3	0,0
D	0,0	1,1

Suppose the state space is given by $\{\alpha, \beta, \gamma, \delta, \epsilon, \zeta\}$. Assume the following information partitions:

$$\mathcal{P}^1 = \{\{\alpha, \beta\}, \{\gamma, \delta\}, \{\epsilon, \zeta\}\}$$
$$\mathcal{P}^2 = \{\{\alpha, \gamma\}, \{\beta, \epsilon\}, \{\delta, \zeta\}\}$$

The actions are given by

- $\sigma^1(\alpha) = \sigma^1(\beta) = U$
- $\sigma^1(\gamma) = \sigma^1(\delta) = \sigma^1(\epsilon) = \sigma^1(\zeta) = D$
- $\sigma^2(\alpha) = \sigma^2(\gamma) = L$
- $\sigma^2(\beta) = \sigma^2(\epsilon) = \sigma^2(\delta) = \sigma^2(\zeta) = R$

The beliefs (conjectures) are given by

- $\mu^1(\alpha) = \mu^1(\beta) = \mu^1(\gamma) = \mu^1(\delta) = (\frac{1}{3}, \frac{2}{3})$
- $\mu^1(\epsilon) = \mu^1(\zeta) = (0, 1)$
- $\mu^2(\alpha) = \mu^2(\gamma) = \mu^2(\beta) = \mu^2(\epsilon) = (\frac{1}{4}, \frac{3}{4})$
- $\mu^2(\delta) = \mu^2(\zeta) = (0, 1)$

a) Are the conditions of Proposition 2 in Section 1.10 of the notes satisfied at α? What can you conclude about players' conjectures being a Nash Equilibrium? What do the players actually play at this state?

Answer:

i knows j's belief at ω if $P^i(\omega) \subset \{\omega' : \mu^j(\omega) = \mu^j(\omega')\}$. Let's start with player 1. $P^1(\alpha) = \{\alpha, \beta\}$. Since $\mu^2(\alpha) = \mu^2(\omega) = (\frac{1}{4}, \frac{3}{4})$, it follows that 1 knows 2's belief at α. Now let's look at player 2. $P^2(\alpha) = \{\alpha, \gamma\}$. Since $\mu^1(\alpha) = \mu^1(\gamma) = (\frac{1}{3}, \frac{2}{3})$, it follows that 2 knows 1's belief at α. Thus, players' conjectures are mutual knowledge.

Now let's check that players' beliefs are consistent with their knowledge. For player i, this means that the support of $\mu^i(\omega)$ is a subset of $\{a^{-i} \in A^{-i} : \sigma^{-i}(\omega') = a^{-i}$ for some $\omega' \in P^i(\omega)\}$. Let's start with player 1. The support of $\mu^1(\alpha)$ is $\{L, R\}$. $L = \sigma^2(\alpha)$ and $R = \sigma^2(\beta)$. Since α and β are both in $P^1(\alpha)$, player 1 has beliefs consistent with his knowledge. Now let's look at player 2. The support of $\mu^2(\alpha)$ is $\{U, D\}$. $U = \sigma^1(\alpha)$ and $D = \sigma^1(\gamma)$. Since α and γ are both in $P^2(\alpha)$, player 2 also has beliefs consistent with his knowledge.

i knows that j is rational at ω if for any $\omega' \in P^i(\omega)$, $\sigma^j(\omega')$ maximizes j's utility against $\mu^j(\omega')$. Let's start with player 1. $P^1(\alpha) = \{\alpha, \beta\}$. At α, player 2 plays L against the belief $(\frac{1}{4}, \frac{3}{4})$. This is rational. At β, player 2 plays R against the belief $(\frac{1}{4}, \frac{3}{4})$. This is also rational. Hence, 1 knows that 2 is rational. Now let's look at player 2. $P^2(\alpha) = \{\alpha, \gamma\}$. At α, player 1 plays U against the belief $(\frac{1}{3}, \frac{2}{3})$. This is rational. At γ, player 1 plays D against the belief $(\frac{1}{3}, \frac{2}{3})$. This is also rational. Hence, 2 knows that 1 is rational. Thus, players' rationality is mutual knowledge.

Since the conditions of Proposition 2 are satisfied, we can conclude that the players' conjectures $\left((\frac{1}{3}, \frac{2}{3}), (\frac{1}{4}, \frac{3}{4})\right)$, form a Nash Equilibrium. Player 1 plays U, while Player 2 plays L.

b) Are the conditions of Proposition 2 in Section 1.10 of the notes satisfied at ζ? What can you conclude about players' conjectures being a Nash Equilibrium? What do the players actually play at this state?

Answer: The conditions of Proposition 2 are not satisfied at ζ. In particular, $P^1(\zeta) = \{\epsilon, \zeta\}$. $\mu^2(\zeta) = (0, 1) \neq \mu^2(\epsilon) = (\frac{1}{4}, \frac{3}{4})$. Thus, player 1 does not know player 2's conjecture. Likewise, player 2 does not know player 1's conjecture.

The theorem is not an if and only if statement, so we cannot conclude anything about the conjectures forming a Nash equilibrium from the observation that the conditions of the theorem are not satisfied. In this example, the players' conjectures at state ζ do not form a Nash equilibrium. In principle, it's possible to construct examples where the conditions of the theorem are not satisfied but the conjectures still form a Nash equilibrium.

At state ζ, player 1 plays D while player 2 plays R.

Question 2 (Dynamic Games)

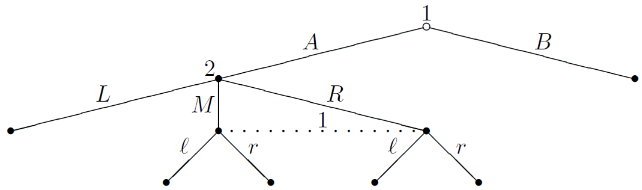
Consider the following two player extensive form game. There are two players: Firm E (Entrant) and Firm I (Incumbent). Firm E chooses whether to enter a market or not. If it does not enter, the payoffs are 0 to Firm E and 2 to Firm I (Firm I gets all the profit). If Firm E enters, Firm I has two choices: to accommodate or fight. If Firm I accommodates, it gets a payoff of 1 while firm E gets a payoff of 2. If Firm I fights, Firm E gets a payoff of -3 and Firm I gets a payoff of -1.

(a) What are the Nash Equilibria?

(b) Argue that the Nash Equilibrium is not a sensible prediction for this game.

Answer: See class notes.

Homework 7



1. For this question, use the extensive form game shown above. Find the behavioral strategy of player 1 that is equivalent to her mixed strategy in which she plays (B, r) with probability 0.4, (B, l) with probability 0.1, and (A, l) with probability 0.5. Show formally that the mixed strategy and the behavioral strategy that you found are equivalent, i.e. that they induce the same probability distribution on Z for any pure strategy of player 2.

Answer: The behavioral strategy must specify two distributions $((p, 1-p), (q, 1-q))$, one for each information set of Player 1.

$$(1-q) = \frac{\text{probability of reaching the first information set and choosing } B \text{ with s}}{\text{probability of reaching the first information set with s}} = 0.4 + 0.1 = 0.5.$$

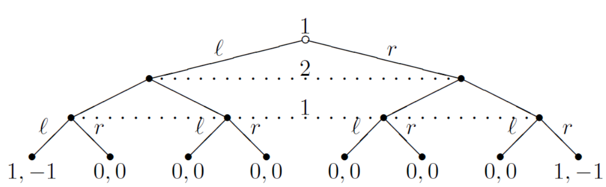
$$p = \frac{\text{probability of reaching the second information set choosing } l \text{ with } s}{\text{probability of reaching the second information set with s}} = \frac{0.5}{0.5} = 1.$$

If player 2 plays L, the implied distribution on final nodes is $(0.5, 0, 0, 0, 0, 0.5)$ with either the mixed or the behavioral strategy.

If player 2 plays M, the implied distribution on final nodes is $(0, 0.5, 0, 0, 0, 0.5)$ with either the mixed or the behavioral strategy.

If player 2 plays M, the implied distribution on final nodes is $(0, 0, 0, 0.5, 0, 0.5)$ with either the mixed or the behavioral strategy.

Hence, the strategies are indeed equivalent.



2. For this question, use the extensive form game shown above. Show that the highest expected payoff player 1 can guarantee with a behavioral strategy is $1/4$, while there is a mixed strategy that guarantees him an expected payoff of $1/2$.

Answer: Consider a behavioral strategy $((p_1, 1-p_1), (p_2, 1-p_2))$ that Player 1 plays against $(q, 1-q)$, a mixed or behavioral strategy of the other player. Player 1's payoff is $p_1 p_2 q + (1-p_1)(1-p_2)(1-q)$. To make it irrelevant what Player 2 plays, it needs to be the case that $p_1 p_2 = (1-p_1)(1-p_2)$. Thus, $\frac{p_1}{1-p_1} = \frac{1-p_2}{p_2}$. It follows that $p_2 = (1-p_1)$ and that Player 1's payoff is $p_1(1-p_1)$. Maximize this function to get $p_1 = \frac{1}{2}$ Hence, $p_2 = \frac{1}{2}$ and player's 1's payoff from the behavioral strategy is $1/4$.

If Player 1 plays a mixed strategy, his payoff is $s(l,l)q + s(r,r)(1-q)$. To make it irrelevant what Player 2 plays, player 1 should set $s(l,l) = s(r,r)$. To maximize his payoff subject to this constraint, he should set the probabilities on all other pure strategies equal to 0. His payoff then is $1/2$.

3. Prove formally that any game satisfying perfect recall is linear.

Answer: Assume an EFG has perfect recall for all i. Suppose by contradiction there exist $x_1 \in u$, $x_2 \in u$, $x_1 \neq x_2$ s.t. $x_1, x_2 \in P(z)$ for some $z \in Z$.

Assume without loss of generality that $x_2 \succsim x_1$. Let a be the action that leads from x_2 to x_1. Since x_1 and x_2 are in the same information set, by perfect recall, there exists another node x_3 such that a leads from x_3 to x_2.

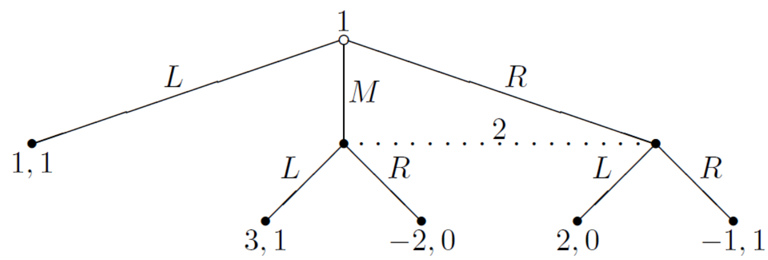
We can hence construct a chain

$$x_n \to^a x_{n-1} \to^a \dots \to^a x_3 \to^a x_2 \to^a x_1.$$

Since the game is finite, eventually the x's have to start repeating. But then we will have $x_n \succsim x_k$ and $x_k \succsim x_n$ for some $x_n \neq x_k$, which is a contradiction with the order being antisymmetric.

4. Finish the argument started in class that a subgame perfect Nash equilibrium exists in any extensive form game of perfect recall.

Answer: See class notes.



5. Find all the sequential equilibria of the game above.

Answer: Find all the sequential equilibria of the game above.

Solution: Let (α, β, γ) denote the strategy of player 1 and $(p, 1-p)$ the strategy of player 2.

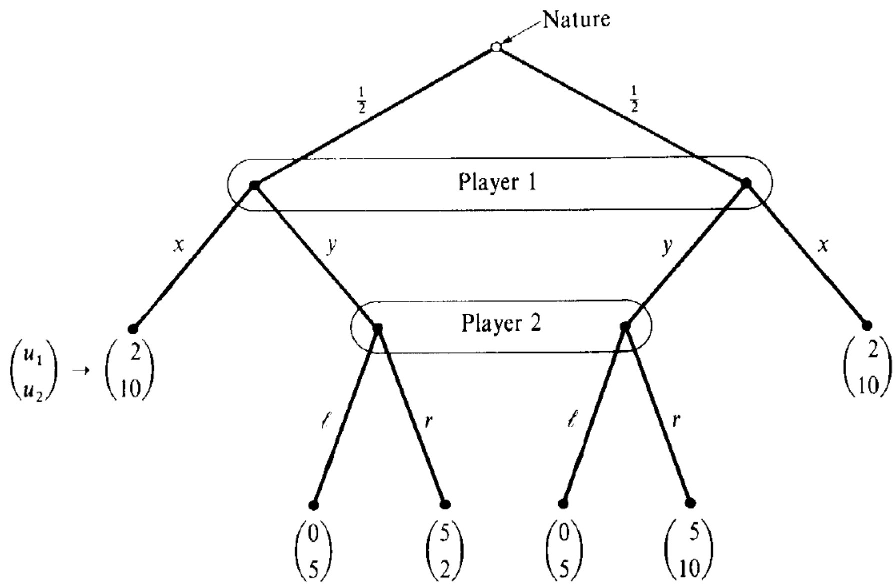
If $\beta > \gamma$, then player 2 chooses L ($p = 1$), and hence player 1 chooses M ($\beta = 1$). (L,M) is indeed a sequential equilibrium strategy profile. It is supported by Player 2's belief that player 1 chooses M.

If $\beta < \gamma$, then player 2 chooses R, and hence player 1 chooses L. This contradicts the assumption that $0 < \gamma$.

If $\beta = \gamma > 0$, then it must be that $3p - 2(1-p) = 2p - (1-p) \Leftrightarrow p = 1/2$. If $p = 1/2$, player 1 should choose L with probability 1, which contradicts that $\gamma > 0$.

If $0 = \beta = \gamma$, then p must be such that $1 \geq 3p - 2(1-p)$ and $1 \geq 2p - (1-p)$. Thus, $3/5 \geq p \geq 0$.

- Any $p \in (0, 3/5]$ can be supported by the belief $(1/2, 1/2)$, which is consistent with the strategy profile $((1,0,0),(p,1-p))$.
- $p = 0$ can be supported by the belief that player 1 is more likely to choose R than M, which is again consistent with $((1,0,0),(p,1-p)) = ((1,0,0),(0,1))$.



6. Find all the sequential equilibria of the game above.

Answer: In this game, any belief derived from a fully mixed strategy gives a distribution $(1/2, 1/2)$ over the nodes in Player 2's information set. Given this, player 2 will chose r and player 1 will chose y. This is the unique sequential equilibrium.