Homework 1

1. Consider the following game. There are two players, Mr. A and Mr. B. The two players are separated and cannot communicate. They are supposed to meet in New York City at noon for lunch but have forgotten to specify where. Each must decide where to go (each can make only one choice). If they meet each other, they get to enjoy each other's company. They each attach a monetary value of 100 dollars to the other's company (their payoffs are each 100 dollars if they meet, 0 if they do not). Suppose there are two meeting places: Grand Central Station and the Empire State Building. Draw a normal form representation for the game.

Answer:

	GCT	EST
GCT	1,1	0,0
EST	0,0	1,1

2. Draw a normal form representation of rock, paper, scissors. Assume that the winning player has to play the losing player 1 dollar.

Answer:

	R	P	S
R	0,0	-1,1	1,-1
P	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

3. In a game where player i has N information sets indexed $n = 1, ..., N$ and M_n possible actions at information set n, how many strategies does player i have?

Answer: $M_1 \times M_2 \times ... \times M_N$

4. There are N firms in an industry. Each can try to convince Congress to give the industry a subsidy. Let h_i denote the number of hours of effort put in by firm i, and let $c_i(h_i) = w_i h_i^2$ denote the cost of effort, with w_i a positive constant. When the effort levels are $(h_1, ..., h_N)$, the value of a subsidy for each firm is $\alpha \sum_i h_i + \beta(\prod_i h_i)$, where $\alpha > 0$ and $\beta > 0$ are constants. There are no other benefits to effort. Consider a game in which the firms decide simultaneously and independently how many hours they will each devote to this effort. Show that each firm has a strictly dominant strategy **if and only** if $\beta = 0$. What is firm i's strictly dominant strategy when this is so?

Answer: Firm i's maximization problem is:

$$\max_{h_i \geq 0} \{\alpha \sum_i h_i + \beta(\prod_i h_i) - w_i h_i^2\}$$

The first order conditions give:

$$h_i* = \frac{\alpha + \beta \prod_{j \neq i} h_j}{2w_i}.$$

$\Rightarrow$: If $\beta = 0$, then h_i* does not depend on other player's choices. It also provides the unique maximum of the function. Hence, it is a strictly dominant strategy.

$\Leftarrow$: If $\beta \neq 0$, then h_i* depends on other player's choices, i.e. is not a strictly dominant strategy. An equivalent way of stating this is: If h_i* is a strictly dominant strategy, then $\beta = 0$.

Homework 2

1. Consider a three player game with three firms $i = 1, 2, 3$. Each firm faces the demand curve $p = a - b(q_1 + q_2 + q_3)$ and per-unit costs of production c. Does iterated elimination of strictly domination actions yield a unique prediction in this game?

Answer: Firm i's maximization problem is:

$$\max_{q_i} \left\{ q_i(a - b(q_i + q_j + q_k)) - cq_i \right\}.$$

The first order conditions give:

$$q_i* = \frac{a-c}{2b} - \frac{q_j + q_k}{2}.$$

Since $q_j \geq 0$ and $q_k \geq 0$, $q_i* \leq \frac{a-c}{2b}$. I.e., any choice of a quantity above $\frac{a-c}{2b}$ is strictly dominated.

Now, since this is true for all players, $q_j* \leq \frac{a-c}{2b}$ and $q_k* \leq \frac{a-c}{2b}$. It follows that

$$q_i* \geq \frac{a-c}{2b} - \frac{a-c}{2b}\frac{2}{2} = 0.$$

Thus, the third step of the elimination procedure brings us back to the first step. The set of quantities surviving IESDA is $[0, \frac{a-c}{2b}]$.

2. Prove formally that if $(R^1, ..., R^N)$ and $(T^1, ..., T^N)$ are rationalizable, then $(R^1 \cup T^1, ..., R^N \cup T^N)$ is rationalizable, as well.

Answer: From the class notes, we get the definition of what it means for the set $(R^1 \cup T^1, ..., R^N \cup T^N)$ to be rationalizable. This means that:

- For all j, $R^j \cup T^j \subset A^j$
- $\forall j, b_j \in R^j \cup T^j, \quad \exists \mu(b^j) \in \Delta(R^{-j} \cup T^{-j})$ s.t. $u^j(b^j, \mu(b^j)) \geq u^j(a^j, \mu(b^j)) \quad \forall a^j \in A^j$.

Take any $j, b_j \in R^j \cup T^j$. Then, either $b^j \in R^j \subset A^j$ or $b^j \in T^j \subset A^j$. Either way, $b^j \in A^j$, so the first part of the definition above is satisfied.

If $b^j \in R^j$, then (by rationalizability) $\exists \mu(b^j) \in \Delta(R^{-j}) \subset \Delta(R^{-j} \cup T^{-j})$ s.t. $u^j(b^j, \mu(b^j)) \geq u^j(a^j, \mu(b^j)) \quad \forall a^j \in A^j$.

If $b^j \in T^j$, then (by rationalizability) $\exists \mu(b^j) \in \Delta(T^{-j}) \subset \Delta(R^{-j} \cup T^{-j})$ s.t. $u^j(b^j, \mu(b^j)) \geq u^j(a^j, \mu(b^j)) \quad \forall a^j \in A^j$.

Either way, $\exists \mu(b^j) \in \Delta(R^{-j} \cup T^{-j})$ s.t. $u^j(b^j, \mu(b^j)) \geq u^j(a^j, \mu(b^j)) \quad \forall a^j \in A^j$, so the second part of the definition is satisfied too.

3. (a) Argue that if a player has two weakly dominant strategies, then for every strategy choice of his opponents, the two strategies yield him equal payoffs.

Answer: We discussed the issues with this in class.

(b) Provide an example of a two player game in which a player has two weakly dominant strategies but his opponent prefers that he play one of them rather than the other.

Answer: We discussed the issues with this in class.

4. Consider the following auction (known as a second-price, or Vickrey, auction). An object is auctioned off to N bidders. Bidder i's valuation of the object (in monetary terms) is v_i. The auction rules are that each player submit a bit (a nonnegative number) in a sealed envelope. The envelopes are then opened, and the bidder who has submitted the highest bid gets the object but pays the auctioneer the amount of the *second-highest* bid. If more than one bidder submits the highest bid, each gets the object with equal probability. Show that submitting a bid of v_i with certainty is a weakly dominant strategy for bidder i.

Answer: We discussed this in class.
5. Show that the set of mixed strategies S is nonempty, compact, and convex.

Answer: We skip the solution because it is straightforward.

6. Show that for every s, $BR^i(s)$ is closed, convex, nonempty, and equal to the mixed strategies concentrated on $PBR^i(s)$.

Answer: We will prove the last part of the claim, since the others are straightforward.

We need to show that $BR^i(s) = \Delta(PBR^i(s)$.

To show this, we need to show that both sets are subsets of each other, i.e. that $BR^i(s) \subset \Delta(PBR^i(s)$ and $\Delta PBR^i(s) \subset BR^i(s)$.

First, let's show that $BR^i(s) \subset \Delta(PBR^i(s)$. Let $s^i \in BR^i(s)$. Then, by definition,

$$u^i(s^i, s^{-i}) \geq u^i(a^i, s^{-i}) \quad \forall a^i \in A^i.$$

It directly follows from this that

$$u^i(s^i, s^{-i}) \geq u^i(\hat{s}^i, s^{-i}) \quad \forall \hat{s}^i \in S^i.$$

Assume by contradiction that $s^i \notin \Delta PBR^i(s)$. Then it puts a positive probability on some action $a^i \notin PBR^i(s)$. Since $a^i \notin PBR^i(s)$, there is some other action $b^i \in A^i$ such that

$$u^i(b^i, s^{-i}) > u^i(a^i, s^{-i}).$$

Now, consider a mixed strategy $\hat{s}^i$ that's identical to s^i expect it shifts the probability that s^i puts on a^i to b^i. Then,

$$u^i(\hat{s}^i, s^{-i}) > u^i(s^i, s^{-i}),$$

which is a contradiction. Thus, $s^i \in \Delta(PBR^i(s))$.

Now, let's show that $\Delta(PBR^i(s)) \subset BR^i(s)$. To this end, let $s^i \in \Delta(PBR^i(s))$. Then, for any $\hat{a}^i \in PBR^i(s)$,

$$u^i(s^i, s^{-i}) = u(\hat{a}^i, s^{-i})$$

By definition of $PBR^i(s)$,

$$u^i(s^i, s^{-i}) = u(\hat{a}^i, s^{-i}) \geq u^i(a^i, s^{-i}) \quad \forall a^i \in A^i.$$

Thus, $s^i \in BR^i(s)$.

Homework 3

1. Argue that (a_2, b_2) is the unique Nash equilibrium of the following game:

	b_1	b_2	b_3	b_4
a_1	0,7	2,5	7,0	0,1
a_2	5,2	3,3	5,2	0,1
a_3	7,0	2,5	0,7	0,1
a_4	0,0	0,-2	0,0	10,-1

Answer: For this question, let $(\beta_1, \beta_2, \beta_3)$ denote the mixed strategy of the column player.

First, consider the case where a_1 and a_3 are both played with a positive probability by player 1. Then $\beta_1 = \beta_3$. This implies that player 1 gets a strictly better payoff from a_2 than from a_1 or a_2, a contradiction.

Now, consider the case where both a_1 and a_2 are played with a positive probability. Then $\beta_3 = 0$. This implies that player 1 gets a strictly better payoff from a_2 than from a_1, a contradiction.

Likewise, the case where a_2 and a_3 are both played with a positive probability leads to a contradiction. This implies that the Nash Equilibrium is in pure strategies, and the unique mixed strategy Nash is (a_2, b_2).

2. Find all the perfect equilibria of the following game:

	A	B	C
A	0,0	0,0	0,0
B	0,0	1,1	2,0
C	0,0	0,2	2,2

Answer: Consider an $\epsilon-$perturbation of the game above. Player 1's payoff from A is 0. His payoff from B is at least $1\epsilon_B^2 + 2\epsilon_C^2$. Similarly, his payoff from C is at least $2\epsilon_C^2$. Since the payoff from A is zero, and the other payoffs are strictly positive, he will never play A in a perturbed game. Likewise, Player 2 will never play A in a perturbed game. It follows that finding perfect equilibria of the game above is equivalent to finding perfect equilibria of the following smaller game:

	B	C
B	1,1	2,0
C	0,2	2,2

Let $\epsilon(n)$ be any sequence of ϵ's converging to the zero vector s.t. $\epsilon_j^i(n) \geq 0$ for all i, j, n and $\sum_j \epsilon_j^i(n) < 1$ for all i, n. Consider an $\epsilon(n)$-perturbation of the gave above.

For all n, given a strategy $(q, 1-q)$ of Player 2, Player 1's payoff from A is $q + 2(1-q)$, while has payoff from B is $2(1-q)$. Since $q > 0$ and $1-q > 0$, the payoff from A is strictly greater. Hence, player 1 will put the smallest probability he can on action B, which is ϵ^1_B. Likewise, Player 2 will put probability ϵ^2_B on his action B. It follows that the equilibrium of the $\epsilon(n)-$perturbed game is

$$((1-\epsilon^1_B, \epsilon^1_B), (1-\epsilon^2_B, \epsilon^2_B)).$$

This converges to $((1,0),(1,0))$ as $\epsilon(n) \to 0$. Hence, (B,B) is the only perfect equilibrium of the game above.

3. Show that the set $NE(G^\epsilon)$ is nonempty for any ϵ s.t. $\epsilon^i_j > 0 \quad \forall i,j$ and $\sum_j \epsilon^i_j < 1 \quad \forall i$.

Hint: Define a correspondence $BR^i_\epsilon(s)$ from S_ϵ to S_ϵ and show that the conditions of Kakutani's Fixed Point Theorem are satisfied.

Answer: Define

$$BR^i_\epsilon(s) = \{s^i \in S^i_\epsilon : u(s^i, s^{-i}) \geq u(t^i, s^{-i}) \quad \forall t^i \in S^i_\epsilon\}.$$

Note this is a correspondence from S_ϵ to S_ϵ. To verify the condition of KFPT, we need to show that

1. S_ϵ is non-empty, closed, convex.
2. BR^i_ϵ is non-empty valued, convex.
3. BR^i_ϵ is upperhemicontinuous.

Clearly, S^i_ϵ is nonempty. It is closed because weak inequalities preserve limits. It is also convex since given any two elements $t^i, s^i \in S^i_\epsilon$ and $\lambda \in (0,1)$, we will also have $\lambda s^i + (1-\lambda)t^i \in S^i_\epsilon$. It follows that $S_\epsilon = \times_i S^i_\epsilon$ is also non-empty valued, closed, and convex.

As we showed before, the set $PBR^i(s)$ is non-empty valued for any s because the game is finite. $BR^i_\epsilon(s)$ is nonempty because given any $a^i \in PBR^i(s)$, the element of S^i_ϵ that puts the biggest probability possible on a^i is in $BR^i_\epsilon(s)$. $BR^i_\epsilon(s)$ is convex because $u^i(\cdot, s^{-i})$ is a linear function in player i's strategy.

It remains to show that $BR^i_\epsilon(s)$ is u.h.c. Let $s(n) \to s$ s.t. $s(n) \in S_\epsilon \quad \forall n$, $t^i_n \in BR^i_\epsilon(s(n)) \quad \forall n$. If $t^i(n) \to t^i$, then $t^i \in BR^i_\epsilon(s)$.

4. (a) A mixed strategy profile is undominated if no player is using a weakly dominated strategy. Show that if s is a perfect equilibrium, then it is also undominated.

Answer. I'll spare you the technical details for this one. Consider any perturbation of the game. The player will put the smallest possible probability on the weakly dominated action (in favor of the action that weakly dominates it). So, any equilibrium of the perturbed game will converge to something that puts the smallest possible probability on the weakly dominated action.

(b) Is the converse true? That is, is an undominated Nash equilibrium perfect? Prove the statement or come up with a counter-example.

Answer: See class notes for the counter-example.

5. Find all the mixed strategy NE of this game, where $0 < \gamma < 1$:

	A	B	C
A	γ, γ	1,-1	-1, 1
B	-1, 1	γ, γ	1, -1
C	1,-1	-1, 1	γ, γ

Answer: Let $(p_1, p_2, 1 - p_1 - p_2)$ denote the mixed strategy of Player 1 and $(q_1, q_2, 1 - q_1 - q_2)$ the mixed strategy of Player 2.

- Assume $p_1 > 0$. Then, it must be the case that $q_2 > 0$. (If $q_2 = 0$, then the payoff from A is $\gamma q_1 - (1 - q_1)$ while the payoff from C is $q_1 + \gamma(1 - q_1)$. Clearly, the payoff from C is strictly greater, which means that A should not be played with a positive probability.)
- Assume $q_2 > 0$. Then, by a similar argument, $1 - p_1 - p_2 > 0$.
- Assume $1 - p_1 - p_2 > 0$. Then, $q_1 > 0$.
- Assume $q_1 > 0$. Then, $p_2 > 0$.
- Assume $p_2 > 0$. Then, $1 - q_1 - q_2 > 0$.
- Assume $1 - q_1 - q_2 > 0$ Then, $p_1 > 0$.

Thus, regardless of what action we assume to be played with a strictly positive probability, we get that all actions of both players are played with a strictly positive probability.

This gives a system of 4 linear inequalities with 4 unknowns (p_1, p_2, q_1, q_2) with the only mixed strategy equilibrium $((1/3, 1/3, 1/3), (1/3, 1/3, 1/3))$ as the solution.

6. Consider the following game:

	A	B
A	2,2	-100,0
B	0,-100	1,1

Find the unique mixed strategy NE, and argue that it is not ESS.

Answer: The unique mixed strategy NE is given by $((q, 1-q), (q, 1-q))$, where $2q - 100(1-q) = (1-q)$. Thus, $q = \frac{101}{103}$. The NE payoff is given by $\frac{2}{103}$.

To show that it is not ESS, consider an $\epsilon-$invasion of a player who always plays A.

- $u(NE) = (1-\epsilon)\frac{2}{103} + \epsilon\frac{101}{103}2$;
- $u(A) = (1-\epsilon)\frac{2}{103} + \epsilon 2$.

Hence, for any ϵ, the utility of the A type is greater.

7. Is a pure strategy NE always strict? Why or why not?

Answer: We did this class.

8. (a) In a Hawk-Dove game discussed in class with $c > 1$, argue that the unique symmetric mixed strategy NE will survive an $\epsilon-$invasion of doves.

Answer: This is straightforward.

(b) Find all the ESS in a Hawk-Dove game discussed in class with $c < 1$ or argue that none exist.

Answer: This is straightforward.